ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC$ the following relationship holds:

$$\sum h_a^3 \leq \sum r_a^3$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tapas Das-India*

$Lemma: \ h_a^2 \leq r_b r_c$

$$Proof: h_a^2 \leq r_b r_c \ or \frac{4F^2}{a^2} \leq \frac{F^2}{(s-b)(s-c)} \ or\ 4(s-b)(s-c) \leq a^2$$

$$or\ 4(s-b)(s-c) \leq \left((s-b)+(s-c)\right)^2 \ true$$

$$since\ \left((s-b)+(s-c)\right)^2 \overset{AM-GM}{\geq} \left(2\left((s-b)(s-c)\right)^{\frac{1}{2}}\right)^2 = 4(s-b)(s-c)$$

$$\sum r_a^3 = \left(\sum r_a\right)^3 - 3\left(\left(\sum r_a\right)\left(\sum r_a r_b\right) - r_a r_b r_c\right) =$$
$$= (4R+r)^3 - 3\left((4R+r)s^2 - s^2 r\right) = (4R+r)^3 - 12Rs^2 \ (1)$$

$$\sum r_b^2 r_c^2 = \left(\sum r_b r_c\right)^2 - 2r_a r_b r_c \left(\sum r_a\right) = s^4 - 2s^2 r(4R+r) \overset{Gerretsen}{\leq}$$
$$\leq s^2(4R^2 + 4Rr + 3r^2 - 8Rr - 2r^2) = s^2(2R-r)^2 \ \ (2)$$

$$\sum h_a^3 \overset{Lemma}{\leq} \sum (r_b r_c)\sqrt{r_b r_c} \overset{CBS}{\leq} \sqrt{\left(\sum (r_b r_c)\right)\left(\sum r_b^2 r_c^2\right)} \overset{(2)}{\leq}$$
$$\leq \sqrt{s^2 s^2 (2R-r)^2} = s^2(2R-r) \ (3)$$

$$from\ (1)\&(2)\ we\ need\ to\ \ show\ (4R+r)^3 - 12Rs^2 \geq s^2(2R-r)\ or$$
$$(4R+r)^3 \geq \ s^2(14R-r) or\ \ (4R+r)^3 \overset{Gerretsen}{\geq} (4R^2+4Rr+3r^2)(14R-r) or$$

$$(4x+1)^3 \overset{\frac{R}{r}=x>2}{\geq} (4x^2+4x+3)(14x-1)\ or$$
$$64x^3 + 48x^2. +12x + 1 \geq 56x^3 + 52x^2 + 38x - 3\ or$$
$$4x^3 - 2x^2 - 13x + 2 \geq 0$$

$$or (x-2)(4x^2+6x-2) \geq 0\ true\ as\ x \geq 2$$

$$Equality\ holds\ for\ a = b = c$$