ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$129r^3 - 6R^3 \le \sum r_a^3 \le 82R^3 - 575r^3$$

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$$\sum r_a^3 = \left(\sum r_a\right)^3 - 3\left(\left(\sum r_a\right)\left(\sum r_a r_b\right) - r_a r_b r_c\right) =$$
$$= (4R+r)^3 - 3\left((4R+r)s^2 - s^2 r\right) = (4R+r)^3 - 12Rs^2 \ (1)$$

$$Now\ from (1) \sum r_a^3 \overset{Euler\ \&\ Mitrinovic}{\le} (4R+r)^3 - 12R(16Rr - 5r^2) =$$
$$= (64R^3 + 48R^2 r + 12Rr^2 + r^3) - 12(2r)27r^2 =$$
$$= (64R^3 + 48R^2 r + 12Rr^2 + r^3) - 648r^3$$

$$We\ need\ to\ show\ that:$$

$$(64R^3 + 48R^2 r + 12Rr^2 + r^3) - 648r^3 \le 82R^3 - 575r^3\ or$$
$$6\,(3R^3 - 8R^2 r - 2Rr^2 + 12r^3) \ge 0\ or$$
$$3x^3 - 8x^2 - 2x + 12 \overset{\frac{R}{r}=x\ge 2}{\ge} 0\ or$$
$$(x-2)(3x^2 - 2x - 6) \ge 0\ or$$
$$(x-2)\left(x(x-2) + 2(x^2-3)\right) \ge 0\ true\ as\ x \ge 2$$
$$From\ (1) \sum r_a^3 \overset{Gerretsen}{\ge} (4R+r)^3 - 12R(4R^2 + 4Rr + 3r^2) =$$
$$= (64R^3 + 48R^2 r + 12Rr^2 + r^3) - 12R(4R^2 + 4Rr + 3r^2)$$

$$We\ need\ to\ show\ that:$$

$$(64R^3 + 48R^2 r + 12Rr^2 + r^3) - 12R(4R^2 + 4Rr + 3r^2) \ge 129r^3 - 6R^3$$

$$or\ 22R^3 - 24Rr^2 - 128r^3 \ge 0\ or$$
$$2(11R^3 - 12Rr^3 - 64r^3) \ge 0\ or$$
$$(R-2r)(11R^2 + 22Rr + 32r^2) \ge 0\ true$$

$$Equality\ holds\ for\ a = b = c$$