ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$1-2\left(\frac{r}{R}\right)^2\le\sum\cos^2 A-\left(\sum\cos A-1\right)^2\le 3-10\left(\frac{r}{R}\right)^2$$

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Solution by Tapas Das-India

$$\sum\cos^2 A-\left(\sum\cos A-1\right)^2=\sum\cos^2 A-\left(\sum\cos A\right)^2+2\sum\cos A-1=$$

$$=2\sum\cos A-2\sum\cos A\cos B-1=2\left(1+\frac{r}{R}\right)-2\left(\frac{s^2+r^2}{4R^2}-1\right)-1=$$

$$=\frac{4Rr-s^2-r^2+6R^2}{2R^2}\ (1)$$

$$From\ (1):\sum\cos^2 A-\left(\sum\cos A-1\right)^2\ =$$

$$=\frac{4Rr-s^2-r^2+6R^2}{2R^2}\overset{Gerretsen}{\le}\frac{4Rr-16Rr+4r^2+6R^2}{2R^2}\overset{Euler}{\le}$$

$$\le\frac{(6R^2-12(2r).r+4r^2)}{2R^2}=\frac{6R^2-20Rr}{2R^2}=3-10\left(\frac{r}{R}\right)^2$$

$$From\ (1):\sum\cos^2 A-\left(\sum\cos A-1\right)^2=\frac{4Rr-s^2-r^2+6R^2}{2R^2}\overset{Gerretsen}{\ge}$$

$$\frac{4Rr-4R^2-4Rr-3r^2-r^2+6R^2}{2R^2}=\frac{2R^2-4r^2}{2R^2}=1-2\left(\frac{r}{R}\right)^2$$

$$Equality\ holds\ iff\ \Delta ABC\ is\ equilateral$$