ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, the following relationship holds :

$$8\sum_{cyc}\sin^2\frac{A}{2}+32\prod_{cyc}\sin\frac{A}{2}+\frac{9}{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}\geq 22$$

Proposed Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\operatorname{cosec}^2\frac{A}{2}=3+\sum_{cyc}\cot^2\frac{A}{2}=3+\sum_{cyc}\frac{s^2}{r_a^2}$$

$$=3+\frac{s^2}{r^2s^4}\left(\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\left(\sum_{cyc}r_a\right)\right)=3+\frac{s^2\left(s^4-2rs^2(4R+r)\right)}{r^2s^4}$$

$$\Rightarrow\sum_{cyc}\operatorname{cosec}^2\frac{A}{2}=\frac{s^2-8Rr+r^2}{r^2}\Rightarrow\left(\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)^2=\sum_{cyc}\operatorname{cosec}^2\frac{A}{2}+$$

$$2\sum_{cyc}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}=\frac{s^2-8Rr+r^2}{r^2}+\frac{2}{\left(\frac{r}{4R}\right)}.\sum_{cyc}\sin\frac{A}{2}\overset{\text{Jensen}}{\leq}$$

$$\frac{s^2-8Rr+r^2}{r^2}+\frac{12R}{r}\overset{\text{Gerretsen}}{\leq}\frac{4R^2+8Rr+4r^2}{r^2}=\frac{4(R+r)^2}{r^2}$$

$$\Rightarrow\sum_{cyc}\operatorname{cosec}\frac{A}{2}\overset{(1)}{\leq}\frac{2R+2r}{r}$$

$$\text{Now, }8\sum_{cyc}\sin^2\frac{A}{2}+32\prod_{cyc}\sin\frac{A}{2}+\frac{9}{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}-22$$

$$=\frac{9}{\left(\prod_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)}-22+8\sum_{cyc}\sin^2\frac{A}{2}+32.\frac{r}{4R}\overset{\text{via }(1)}{\geq}$$

$$\frac{9}{\left(\frac{r}{4R}\right)\left(\frac{2R+2r}{r}\right)}-22+\frac{4(2R-r)}{R}+\frac{8r}{R}=\frac{2\left(2R^2-5Rr+2r^2\right)}{R}=$$

$$\frac{2(2R-r)(R-2r)}{R}\overset{\text{Euler}}{\geq}0\Rightarrow 8\sum_{cyc}\sin^2\frac{A}{2}+32\prod_{cyc}\sin\frac{A}{2}+\frac{9}{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\geq 22\ \forall\ \Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$