ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\sqrt{3}\tan^2\frac{A}{2}+2\tan\frac{B}{2}}\ge\frac{1}{\sqrt{3}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\sqrt{3}\tan\frac{A}{2},\ y=\sqrt{3}\tan\frac{B}{2},\ y=\sqrt{3}\tan\frac{C}{2}\text{ and then :}$$

$$\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\sqrt{3}\tan^2\frac{A}{2}+2\tan\frac{B}{2}}\ge\frac{1}{\sqrt{3}}\Leftrightarrow\sum_{cyc}\frac{\sqrt{3}.\frac{x^2}{3}}{\sqrt{3}.\frac{x^2}{3}+\frac{2y}{\sqrt{3}}}\ge 1$$

$$\Leftrightarrow\sum_{cyc}x^2(y^2+2z)(z^2+2x)\ge\prod_{cyc}(x^2+2y)$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow}\ x^2y^2z^2-4xyz+\sum_{cyc}x^3y^2\ge 0$$

$$\Leftrightarrow 27\prod_{cyc}\tan^2\frac{A}{2}-12\sqrt{3}.\prod_{cyc}\tan\frac{A}{2}+9\sqrt{3}.\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}\ge 0$$

$$\Leftrightarrow 3\sqrt{3}.\frac{r^2}{s^2}+3\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}\boxed{\overset{(*)}{\ge}}\frac{4r}{s}$$

$$\text{Now, }3\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}=3\sum_{cyc}\frac{\tan^3\frac{A}{2}\tan^3\frac{B}{2}}{\tan\frac{B}{2}}\overset{\text{Holder}}{\ge}\frac{3\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^3}{3\sum_{cyc}\tan\frac{A}{2}}=\frac{s}{4R+r}$$

$$\Rightarrow\left(3\sqrt{3}.\frac{r^2}{s^2}+3\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}\right)^2\ge\left(3\sqrt{3}.\frac{r^2}{s^2}+\frac{s}{4R+r}\right)^2$$

$$=\frac{27r^4}{s^4}+\frac{s^2}{(4R+r)^2}+\frac{6\sqrt{3}r^2}{s(4R+r)}\overset{\text{Trucht or Doucet}}{\ge}\frac{27r^4}{s^4}+\frac{s^2}{(4R+r)^2}+\frac{18r^2}{(4R+r)^2}$$

$$\overset{?}{\ge}\frac{16r^2}{s^2}\Leftrightarrow s^4(s^2+18r^2)+27r^4(4R+r)^2-16r^2s^2(4R+r)^2\overset{?}{\ge}0$$

$$\Leftrightarrow s^6+18r^2s^4-r^2s^2(256R^2+128Rr+16r^2)+27r^4(4R+r)^2\boxed{\underset{(**)}{\overset{?}{\ge}}}0$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\ge}(16Rr+13r^2)s^4-r^2s^2(256R^2+128Rr+16r^2)$$

$$+27r^4(4R+r)^2 \overset{\text{Gerretsen}}{\geq} \begin{pmatrix}(16Rr+13r^2)(16Rr-5r^2)\\ -r^2(256R^2+128Rr+16r^2)\end{pmatrix}s^2+27r^4(4R+r)^2$$

$$=-81r^4s^2+27r^4(4R+r)^2=27r^4\left((4R+r)^2-3s^2\right)\overset{\text{Trucht or Doucet}}{\geq}0$$

$$\Rightarrow (**) \text{ is true} \therefore \left(3\sqrt{3}.\frac{r^2}{s^2}+3\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}\right)^2 \geq \frac{16r^2}{s^2} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{\tan^2\frac{A}{2}}{\sqrt{3}\tan^2\frac{A}{2}+2\tan\frac{B}{2}} \geq \frac{1}{\sqrt{3}}\ \forall\,\Delta\,ABC,$$

with equality iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=\sqrt{3}\tan\frac{A}{2}, y:=\sqrt{3}\tan\frac{B}{2}, z:=\sqrt{3}\tan\frac{C}{2}$. We have $xy+yz+zx=3$.

The desired inequality is equivalent to

$$\sum_{cyc}\frac{x^2}{x^2+2y}\geq 1.$$

By CBS inequality, we have

$$\sum_{cyc}\frac{x^2}{x^2+2y}\geq\frac{\left(\sqrt{x^3}+\sqrt{y^3}+\sqrt{z^3}\right)^2}{\sum_{cyc}x(x^2+2y)}=\frac{\sum_{cyc}x^3+2\sum_{cyc}(yz)^{\frac{3}{2}}}{\sum_{cyc}x^3+2\sum_{cyc}yz}\geq$$

$$\overset{\text{Power Mean}}{\geq}\frac{\sum_{cyc}x^3+2.3\left(\frac{\sum_{cyc}yz}{3}\right)^{\frac{3}{2}}}{\sum_{cyc}x^3+6}=1.$$

which complete the proof. Equality holds iff ΔABC is equilateral.