ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\sum \sin^2 A}{\sum \sin A \sin B}=\frac{4}{3}\ then\ GI^2\geq 2r^2$$

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Solution by Tapas Das-India

$$\sin A=\frac{a}{2R},\sin B=\frac{b}{2R},\sin C=\frac{c}{2R}$$

$$we\ know\ that\ GI^2=\frac{1}{9}(s^2+5r^2-16Rr)$$

$(Reference: Useful\ Identities\ and\ inequalities\ in\ Geometry,$
$contributor: Samer\ Seraj, Andrew\ Krik, Reda\ Afara, Luis\ Gonzales)$

$$\frac{\sum \sin^2 A}{\sum \sin A \sin B}=\frac{\sum a^2}{\sum ab}=\frac{2(s^2-r^2-4Rr)}{s^2+r^2+4Rr}$$

$$\frac{\sum \sin^2 A}{\sum \sin A \sin B}=\frac{4}{3}\ or\ \frac{2(s^2-r^2-4Rr)}{s^2+r^2+4Rr}=\frac{4}{3}\ or$$

$$6(s^2-r^2-4Rr)=4(s^2+r^2+4Rr)\ or$$

$$2s^2-10r^2-40Rr=0\ or$$

$$s^2=5r^2+20Rr\quad (1)$$

$$We\ need\ to\ show\ GI^2\geq 2r^2\ or\ \frac{1}{9}(s^2+5r^2-16Rr)\geq 2r^2\ or$$

$$(s^2+5r^2-16Rr)\geq 18r^2\ or\ (5r^2+20Rr+5r^2-16Rr)\overset{(1)}{\geq} 18r^2\ or$$

$$4Rr\geq 8r^2\ or\ R\geq 2r\ True(Euler)$$