ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \frac{\tan^2 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \geq \frac{4R + r}{p} \left(\frac{1}{2} \left(\frac{4R + r}{p} \right)^2 - 1 \right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{\tan^2 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} = \sum_{cyc} \frac{\tan^4 \frac{A}{2}}{\tan^2 \frac{A}{2} \tan \frac{B}{2} + \tan^2 \frac{A}{2} \tan \frac{C}{2}} \overset{\text{Bergstrom}}{\geq}$$

$$\geq \frac{\left(\sum_{cyc} \tan^2 \frac{A}{2} \right)^2}{\sum_{cyc} \tan^2 \frac{A}{2} \tan \frac{B}{2} + \sum_{cyc} \tan^2 \frac{A}{2} \tan \frac{C}{2}} =$$

$$= \frac{\frac{1}{p^4} \left((4R + r)^2 - 2p^2 \right)^2}{\left(\sum_{cyc} \tan \frac{A}{2} \right) \left(\sum_{cyc} \tan \frac{A}{2} \tan \frac{B}{2} \right) - 3 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}$$

$$= \frac{\left((4R + r)^2 - 2p^2 \right)^2}{p^4 \left(\frac{4R+r}{p} - \frac{3r}{p} \right)} \overset{?}{\geq} \frac{4R + r}{p} \left(\frac{(4R + r)^2 - 2p^2}{2p^2} \right)$$

$$\Leftrightarrow 2(4R + r)^2 - 4p^2 \overset{?}{\geq} (4R - 2r)(4R + r) \Leftrightarrow p^2 \overset{?}{\leq} 4R^2 + 5Rr + r^2$$

$$\Leftrightarrow \left(p^2 - (4R^2 + 4Rr + 3r^2) \right) - r(R - 2r) \overset{?}{\leq} 0 \rightarrow \text{true}$$

$$\because p^2 - (4R^2 + 4Rr + 3r^2) \overset{\text{Gerretsen}}{\leq} 0 \text{ and } -r(R - 2r) \overset{\text{Euler}}{\leq} 0$$

$$\therefore \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \geq \frac{4R + r}{p} \left(\frac{1}{2} \left(\frac{4R + r}{p} \right)^2 - 1 \right)$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)