ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{2}{3}(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

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$$We\ will\ show\ that:$$

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}\ or$$

$$\sum_{cyc}\left(\frac{a}{b}-\frac{a}{b+c}\right)\geq\frac{3}{2}$$

$$Proof:\sum_{cyc}\left(\frac{a}{b}-\frac{a}{b+c}\right)=\sum_{cyc}\frac{ac}{b(b+c)}=\sum\frac{\left(\sqrt{\frac{ac}{b}}\right)^2}{b+c}\overset{Bergstrom}{\geq}$$

$$\geq\frac{\left(\sqrt{\frac{ac}{b}}+\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}\right)^2}{2(a+b+c)}\overset{\forall x,y,z>0\ (\sum x)^2\geq3\sum xy}{\geq}\frac{3\sum\sqrt{\frac{ac}{b}}\sqrt{\frac{ab}{c}}}{2(a+b+c)}=$$

$$=\frac{3\sum\sqrt{a^2}}{2(a+b+c)}=\frac{3(a+b+c)}{2(a+b+c)}=\frac{3}{2}$$

$$so\ the\ proof\ complete.$$

$$Now\ \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}\geq$$

$$=\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{3}{2}\overset{Nesbitt}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{3}{2}+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{3}{2}=$$

$$=\frac{1}{2}+\frac{3}{2}+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=$$

$$=2+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=\frac{2}{3}\left(3+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=$$

$$=\frac{2}{3}\sum\left(1+\frac{a}{b+c}\right)=\frac{2}{3}\sum\left(\frac{a+b+c}{b+c}\right)=$$

$$=\frac{2}{3}(a+b+c)\sum\frac{1}{b+c}=\frac{2}{3}(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

$$Equality\ if\ a=b=c$$