ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{\cos B\cos C}{\cos(B-C)} \le \cos^2 A+\cos^2 B+\cos^2 C$$

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$$\frac{b\cos B+c\cos C}{a}=\frac{R(\sin 2B+\sin 2C)}{2R\sin A}=$$

$$=\frac{2R\sin(B+C)\cos(B-C)}{2R\sin A}\overset{A+B+C=\pi}{=}\frac{2R\sin(A)\cos(B-C)}{2R\sin A}=$$

$$=\cos(B-C)\quad and\ \cos(B-C)=\frac{b\cos B+c\cos C}{a}=$$

$$=\frac{R(\sin 2B+\sin 2C)}{2R\sin A}=\frac{(\sin 2B+\sin 2C)}{2\sin A}\ (1)$$

$$\sin 2A+\sin 2B+\sin 2C=2\sin(A+B)\cos(A-B)+2\sin C\cos C$$
$$=2\sin(\pi-C)\cos(A-B)+2\sin C\cos C=$$
$$=2\sin C\cos(A-B)+2\sin C\cos C=2\sin C\,(\cos(A-B)+\cos C)=$$
$$=2\sin C\left(\cos(A-B)+\cos\big(\pi-(A+B)\big)\right)=$$
$$=2\sin C\,(\cos(A-B)-\cos(A+B))=4\sin A\sin B\sin C\quad(2)$$

$$\cos(B-C)=\cos B\cos C+\sin B\sin C\quad(3)$$

$$\sum\frac{\cos B\cos C}{\cos(B-C)}\overset{(3)}{=}\sum\left(1-\frac{\sin B\sin C}{\cos(B-C)}\right)\overset{(1)}{=}$$

$$=3-\sum\left(\frac{\sin B\sin C}{\cos(B-C)}\right)\overset{(1)}{=}3-\sum\left(\frac{2\sin A\sin B\sin C}{\sin 2B+\sin 2C}\right)=$$

$$=3-2\sin A\sin B\sin C\sum\left(\frac{1}{\sin 2B+\sin 2C}\right)\overset{Bergstrom}{\le}$$

$$\le 3-2\sin A\sin B\sin C\frac{(1+1+1)^2}{2(\sin 2A+\sin 2B+\sin 2C)}\overset{(2)}{=}$$

$$=3-2\sin A\sin B\sin C\frac{9}{8\sin A\sin B\sin C}=3-\frac{9}{4}=\frac{3}{4}\ (4)$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 3 - \sum \sin^2 A = 3 - \frac{a^2 + b^2 + c^2}{4R^2} \overset{Leibniz}{\geq}$$

$$\geq\ 3 - \frac{9R^2}{4R^2} = 3 - \frac{9}{4} = \frac{3}{4}\ (5)$$

From (4)*and*(5)*we get* $\sum \frac{\cos B \cos C}{\cos(B - C)} \leq \cos^2 A + \cos^2 B + \cos^2 C$

Equality holds $A = B = C$.