ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \sqrt[3]{\frac{3r}{r_a}} \ge \frac{16r}{R} - 5$$

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Solution by Tapas Das-India

$$\sqrt[3]{\frac{3r}{r_a}} = \sqrt[3]{\frac{3r}{r_a}1.1} \overset{GM\ge HM}{\ge} \frac{3}{\frac{1}{\frac{3r}{r_a}} + \frac{1}{1} + \frac{1}{1}} = \frac{3}{\frac{r_a}{3r} + 2} \quad (1)$$

$$\sum \sqrt[3]{\frac{3r}{r_a}} \overset{(1)}{\ge} \sum \frac{3}{\frac{r_a}{3r} + 2} = 3\sum \frac{1^2}{\frac{r_a}{3r} + 2} \overset{Bergstrom}{\ge} \frac{3(1+1+1)^2}{\frac{1}{3r}(r_a + r_b + r_c) + 6} =$$

$$= \frac{27}{\frac{4R+r}{3r} + 6} = \frac{81r}{4R + r + 18r} = \frac{81r}{4R + 19r} \overset{EULER}{\ge} \frac{81r}{4R + \frac{19R}{2}} =$$

$$= 81r \cdot \frac{2}{27R} = \frac{6r}{R} = \frac{16r}{R} - \frac{10r}{R} \overset{EULER}{\ge} \frac{16r}{R} - 10 \cdot \frac{1}{2} = \frac{16r}{R} - 5$$

Equality holds for the equilateral triangle.