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In any Δ ABC, the following relationship holds :

$$\frac{r_a}{a\sin A}+\frac{r_b}{b\sin B}+\frac{r_c}{c\sin C}\le\frac{3R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{a\sin A}+\frac{r_b}{b\sin B}+\frac{r_c}{c\sin C}=\sum_{cyc}\frac{2Rs\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}.a}=\frac{s}{2}\sum_{cyc}\frac{\sec^2\frac{A}{2}}{a}\overset{\text{Chebyshev}}{\le}$$

$$\le\frac{s}{6}.\sum_{cyc}\sec^2\frac{A}{2}.\sum_{cyc}\frac{1}{a}=\frac{s}{6}.\frac{s^2+(4R+r)^2}{s^2}.\frac{s^2+4Rr+r^2}{4Rrs}\overset{\substack{\text{Doucet or Trucht}\\ \text{and}\\ \text{Gerretsen + Euler}}}{\le}$$

$$\le\frac{s}{6}.\frac{\frac{(4R+r)^2}{3}+(4R+r)^2}{s^2}.\frac{s^2+\frac{s^2}{3}}{4Rrs}=\frac{1}{6}.\frac{16}{9}.\frac{(4R+r)^2}{4Rr}\overset{\text{Euler}}{\le}\frac{1}{6}.\frac{16}{9}.\frac{81R^2}{16Rr}$$

$$\therefore\frac{r_a}{a\sin A}+\frac{r_b}{b\sin B}+\frac{r_c}{c\sin C}\le\frac{3R}{2r}\ \forall\,\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$