ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda \geq 0$ then in ΔABC the following relationship holds:

$$\frac{1}{\lambda+\tan A\tan B}+\frac{1}{\lambda+\tan B\tan C}+\frac{1}{\lambda+\tan C\tan A}\leq\frac{3}{\lambda+3}$$

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Solution by Tapas Das-India

$$We\ know\ that\ in\ \Delta ABC\ ,\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

$$Let\ \tan A=\frac{1}{x},\tan B=\frac{1}{y},\tan C=\frac{1}{z}\ then$$

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C\quad can\ be\ writen\ as\ xy+yz+zx=1\ (1)$$

$$\frac{1}{\lambda+\tan A\tan B}+\frac{1}{\lambda+\tan B\tan C}+\frac{1}{\lambda+\tan C\tan A}=$$

$$=\sum\frac{1}{\lambda+\tan A\tan B}=\sum\frac{1}{\lambda+\frac{1}{x}\cdot\frac{1}{y}}=\sum\frac{xy}{\lambda xy+1}=$$

$$=\frac{1}{\lambda}\sum\left(1-\frac{1}{\lambda xy+1}\right)=\frac{3}{\lambda}-\frac{1}{\lambda}\sum\left(\frac{1^2}{\lambda xy+1}\right)\overset{Bergstrom}{\leq}$$

$$\leq\frac{3}{\lambda}-\frac{1}{\lambda}\cdot\frac{(1+1+1)^2}{\lambda(xy+yz+zx)+3}\overset{(1)}{=}\frac{3}{\lambda}-\frac{1}{\lambda}\cdot\frac{9}{\lambda+3}=$$

$$=\frac{1}{\lambda}\left(3-\frac{9}{\lambda+3}\right)=\frac{1}{\lambda}\cdot\frac{3\lambda}{\lambda+3}=\frac{3}{\lambda+3}$$

$$Equality\ holds\ A=B=C=\frac{\pi}{3}$$