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In ΔABC the following relationship holds:

$$2sr\left(7-\frac{2r}{R}\right)\le\sum a(h_b+h_c)\le 6Rs$$

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Solution by Tapas Das-India

$$\sum a(h_b+h_c)=\sum a\left(\frac{ac}{2R}+\frac{ab}{2R}\right)=\frac{1}{2R}\sum a^2(b+c)\overset{Chebyshev}{\le}$$

$$\le\frac{1}{3}\cdot\frac{1}{2R}\left(\sum a^2\right)\left(\sum b+c\right)\overset{Leibniz}{\le}\frac{9R^2}{2R}.2(a+b+c)\frac{1}{3}=3R\cdot 2s=6Rs$$

$$\sum a(h_b+h_c)=\sum a(h_b+h_a+h_c)-\sum ah_a=\left(\sum h_a\right)\left(\sum a\right)-\sum a\cdot\frac{2F}{a}=$$

$$=2s\left(\frac{bc}{2R}+\frac{ca}{2R}+\frac{ab}{2R}\right)-6F=2s\cdot\frac{\sum bc}{2R}-6F=$$

$$=\frac{s}{R}(s^2+r^2+4Rr)-6rs\overset{Gerretsen}{\ge}\frac{s}{R}(16Rr-5r^2+r^2+4Rr)-6rs=$$

$$=\frac{2rs(10R-2r-3R)}{R}=\frac{2rs(7R-2r)}{R}=2sr\left(7-\frac{2r}{R}\right)$$

$$Equality\ holds\ for\ a=b=c$$