ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{h_b + h_c} \geq \sum \frac{1}{r_b + r_c}$$

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$$\sum \frac{1}{h_b + h_c} = \sum \frac{1}{\frac{ac}{2R} + \frac{ab}{2R}} = 2R \sum \frac{1^2}{ab + ac} \overset{Bergstrom}{\geq}$$

$$\geq 2R \frac{(1+1+1)^2}{2\sum ab} \overset{(\sum x)^2 \geq 3\sum xy}{\geq} \frac{9R.3}{(\sum a)^2} = \frac{27R}{(2s)^2} = \frac{27R}{4s^2} \ (1)$$

$$\sum \frac{1}{r_b + r_c} = \frac{\sum (r_a + r_c)(r_a + r_b)}{\prod (r_b + r_c)} = \frac{\sum (r_a^2 + r_a r_b + r_b r_c + r_a r_c)}{\prod (r_b + r_c)} =$$

$$= \frac{\sum r_a^2 + 3\sum r_a r_b}{\prod (r_b + r_c)} = \frac{(\sum r_a)^2 + \sum r_a r_b}{(\sum r_a)(\sum r_a r_b) - \prod r_a} = \frac{(4R + r)^2 + s^2}{(4R + r)s^2 - s^2 r} =$$

$$= \frac{(4R + r)^2 + s^2}{4Rs^2} \overset{Gerretsen}{\leq} \frac{16R^2 + 8Rr + r^2 + 4R^2 + 4Rr + 3r^2}{4Rs^2} =$$

$$= \frac{20R^2 + 12Rr + 4r^2}{4Rs^2} \ (2)$$

$$We\ need\ to\ show \sum \frac{1}{h_b + h_c} \geq \sum \frac{1}{r_b + r_c} \ or$$

$$\frac{27R}{4s^2} \overset{using\ 2\ \&3}{\geq} \frac{20R^2 + 12Rr + 4r^2}{4Rs^2} \ or,$$

$$27R^2 \geq 20R^2 + 12Rr + 4r^2 \ or,$$

$$7R^2 - 12Rr - 4r^2 \geq 0 \ or \ (R - 2r)(7R + 2r) \geq 0 \ true \ (Euler)$$

Equality holds for an equilateral triangle