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In ΔABC the following relationship holds:

$$\frac{1}{R} \le \sum \frac{1}{h_b + h_c} \le \frac{R}{4r^2}$$

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Solution by Tapas Das-India

$$\sum \frac{1}{h_b + h_c} = \sum \frac{1^2}{h_b + h_c} \overset{Bergstrom}{\ge} \frac{(1+1+1)^2}{2\sum h_a} \overset{m_a \ge h_a}{\ge}$$

$$\ge \frac{9}{2\sum m_a} \overset{Leuenberger}{\ge} \frac{9}{2(4R+r)} \overset{Euler}{\ge} \frac{9}{2.\left(4R+\frac{R}{2}\right)} = \frac{1}{R}$$

$$\sum \frac{1}{h_b + h_c} \overset{AM-HM}{\le} \frac{1}{4}\sum\left(\frac{1}{h_b} + \frac{1}{h_c}\right) = \frac{1}{2}\sum\left(\frac{1}{h_b}\right) = \frac{1}{2r} = \frac{R}{2Rr} \overset{Euler}{\le} \frac{R}{2.r.2r} = \frac{R}{4r^2}$$

Equality holds for $a = b = c$