ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{\sec^3 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \geq 4$$

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Solution by Tapas Das-India

$$\tan \frac{B}{2} + \tan \frac{C}{2} = \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} = \frac{\sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} =$$

$$\frac{\sin \frac{B+C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} = \frac{\sin \frac{\pi-A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} = \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \ (1)$$

$$\frac{sec^3 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \overset{(1)}{=} \frac{1}{\cos^3 \frac{A}{2} \frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}} = \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos^4 \frac{A}{2}} \ (2)$$

$$\sum \frac{sec^3 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \overset{(2)}{=} \sum \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos^4 \frac{A}{2}} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{1}{\prod \cos^2 \frac{A}{2}}} =$$

$$= 3\sqrt[3]{\left(\frac{4R}{s}\right)^2} \overset{Mitrinovic}{\geq} 3\sqrt[3]{\left(\frac{4R}{\frac{3\sqrt{3}R}{2}}\right)^2} = 3.\sqrt[3]{\left(\frac{8}{3\sqrt{3}}\right)^2} = 3.\frac{4}{3} = 4$$

$$Equality\ holds\ for\ A = B = C = \frac{\pi}{3}$$