ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum a(h_b+h_c)\le\sum a(r_b+r_c)$$

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$$\sum ar_a=F\sum\frac{a}{s-a}=\frac{F}{(s-a)(s-b)(s-c)}\left(\sum a(s-b)(s-c)\right)=$$
$$=\frac{F}{sr^2}(s^2(a+b+c)-2s(ab+bc+ca)+3abc)$$
$$=\frac{F}{sr^2}(2s^3-2s(s^2+r^2+4Rr)+12Rrs)=\frac{F}{sr^2}(2Rr-r^2)2s=\frac{2s(2Rr-r^2)}{r}\quad(1)$$

$$\sum h_a=\sum\frac{bc}{2R}=\frac{\sum bc}{2R}=\frac{s^2+r^2+4Rr}{2R}\overset{Gerretsen}{\le}$$
$$\le\frac{4R^2+4Rr+3r^2+r^2+4Rr}{2R}=\frac{4(R+r)^2}{2R}=\frac{2(R+r)^2}{R}\ (2)$$

$$\sum a(h_b+h_c)=\sum a(h_a+h_b+h_c)-\sum a\,h_a=$$
$$=\left(\sum a\right)\left(\sum h_a\right)-\sum a\frac{2F}{a}\overset{(2)}{\le}2s.\frac{2(R+r)^2}{R}-6F=\frac{2s}{R}(2(R+r)^2-3Rr)\ (3)$$

$$\sum a(r_b+r_c)=\sum a(r_b+r_a+r_c)-\sum a(r_a)\overset{(1)}{=}$$
$$=\left(\sum a\right)\left(\sum r_a\right)-\frac{2s(2Rr-r^2)}{r}=2s.(4R+r)-2s(2R-r)=2s(2R+r)\ (4)$$

$$We\ need\ to\ show\ \sum a(h_b+h_c)\le\sum a(r_b+r_c)\ or$$
$$2s(2R+r)\overset{(3)\&(4)}{\ge}=\frac{2s}{R}(2(R+r)^2-3Rr)\ or$$

$$2R^2+2Rr\ge 2R^2+4Rr+2r^2-3Rr\ or\ Rr\ge 2r^2\ or$$

$$R\ge 2r\ True\ Euler$$

$$Equality\ holds\ for\ a=b=c$$