ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following inequality holds:

$$\prod\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 2\prod\tan\frac{A}{2}\left(\sum\tan^2\frac{A}{2}+3\right)$$

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Solution by Tapas Das-India

$$We\ know\ that\ \tan\frac{A}{2}=\frac{r_a}{s},\tan\frac{B}{2}=\frac{r_b}{s},\tan\frac{C}{2}=\frac{r_c}{s},and$$

$$\left(\sum\tan^2\frac{A}{2}\right)=\left(\frac{4R+r}{s}\right)^2-2,\prod\tan\frac{A}{2}=\frac{r}{s},\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)=1$$

$$\prod\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=\left(\sum\tan\frac{A}{2}\right)\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)-\prod\tan\frac{A}{2}=\frac{4R+r}{s}-\frac{r}{s}=\frac{4R}{s}$$

$$2\prod\tan\frac{A}{2}\left(\sum\tan^2\frac{A}{2}+3\right)=2.\frac{r}{s}\left(\left(\frac{4R+r}{s}\right)^2-2+3\right)=\frac{2r}{s}\left(\left(\frac{4R+r}{s}\right)^2+1\right)$$

$$We\ need\ to\ show\ \frac{2r}{s}\left(\left(\frac{4R+r}{s}\right)^2+1\right)\leq\frac{4R}{s}\ or$$

$$2Rs^2\geq r(4R+r)^2+rs^2$$

$$or\ using\ Gerretsen$$

$$2R(16Rr-5r^2)\geq r(16R^2+8Rr+r^2)+r(4R^2+4Rr+3r^2)$$

$$or\ 12R^2r-22Rr^2-4r^3\geq 0\ or\ 2r(6R+r)(R-2r)\geq 0\ true\ (Euler)$$

$$Equality\ holds\ for\ A=B=C$$