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In ΔABC the following relationship holds:

$$\sum \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}} \le \frac{s\sqrt{2}}{r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tapas Das-India

$$Let\ \tan\frac{A}{2}=x, \tan\frac{B}{2}=y\ , \tan\frac{C}{2}=z\ .$$

$$We\ \ know\ that\ in\ \Delta\ ABC:$$

$$\sum \tan\frac{A}{2}\tan\frac{B}{2}=1\ or, \sum xy=1\ (1)$$

$$xyz=\ \tan\frac{A}{2}, \tan\frac{B}{2}\tan\frac{C}{2}=\frac{r}{s}\ (2)\ and$$

$$\sum x=\sum \tan\frac{A}{2}=\frac{(4R+r)}{s} \overset{s^2\ge 3r(4R+r)}{\le} \frac{s^2}{3rs}=\frac{s}{3r}\ \ (3)$$

$$\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}=xy-x^2y^2=xy(1-xy) \overset{(1)}{=}$$

$$=xy(xy+yz+zx-xy)=xyz(x+y)\ (4)$$

$$\sum \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}=\sum \frac{x+y}{\sqrt{xy-x^2y^2}} \overset{(4)}{=}$$

$$=\sum \frac{x+y}{\sqrt{xyz(x+y)}}=\frac{1}{\sqrt{xyz}}\sum \sqrt{x+y} \overset{CBS}{\le}$$

$$\le \frac{1}{\sqrt{xyz}}\sqrt{3(2x+2y+2z)}=\frac{1}{\sqrt{xyz}}\sqrt{6(x+y+z)} \overset{(2)\&(3)}{\le} \sqrt{\frac{s}{r}}\sqrt{6\frac{s}{3r}}=\frac{s\sqrt{2}}{r}$$

$$Equality\ holds\ for\ an\ equilateral\ triangle$$

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Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}=\sum_{cyc}\frac{\frac{r_a+r_b}{p}}{\sqrt{\frac{r_ar_b}{p^2}-\frac{r_a^2r_b^2}{p^4}}}=\sum_{cyc}\frac{\frac{4R}{p}\cdot\frac{p(p-c)}{ab}}{\sqrt{\frac{p(p-c)}{p^2}-\frac{p^2(p-c)^2}{p^4}}}=$$

$$=\sum_{cyc}\frac{\frac{4R}{p}\cdot\frac{pc(p-c)}{4Rrp}}{\sqrt{\frac{p(p-c)}{p^2}\cdot\left(1-\frac{p(p-c)}{p^2}\right)}}=\sum_{cyc}\frac{\frac{c(p-c)}{rp}}{\sqrt{\frac{p(p-c)}{p^2}\cdot\frac{pc}{p^2}}}=\sum_{cyc}\frac{\frac{c(p-c)}{rp}\cdot p}{\sqrt{c(p-c)}}=\frac{1}{r}\cdot\sum_{cyc}\sqrt{c(p-c)}\le$$

$$\overset{CBS}{\le}\frac{1}{r}\cdot\sqrt{\sum_{cyc}c}\cdot\sqrt{\sum_{cyc}(p-c)}=\frac{1}{r}\cdot\sqrt{2p.p}\therefore\sum_{cyc}\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}}\le\frac{p.\sqrt{2}}{r}$$

$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC$ is equilateral (QED)