ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$3 \le \frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C} \le 3\left(\frac{R}{2r}\right)^3$$

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$$\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}=\sum_{cyc}\frac{bc}{(2R\sin A)a}=\sum_{cyc}\frac{bc}{a^2}=\frac{1}{16R^2r^2s^2}\cdot\sum_{cyc}b^3c^3$$

$$=\frac{(s^2+4Rr+r^2)^3-3(4Rrs)\left(2s(s^2+2Rr+r^2)\right)}{16R^2r^2s^2}\overset{?}{\le}3\left(\frac{R}{2r}\right)^3$$

$$\Leftrightarrow rs^6-(12Rr^2-3r^3)s^4-\left(6R^5-3r^5\right)s^2+r^4(4R+r)^3\ \boxed{\underset{(*)}{\overset{?}{\le}}}\ 0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\le}\left(r(4R^2+4Rr+3r^2)-(12Rr^2-3r^3)\right)s^4$$
$$-\left(6R^5-3r^5\right)s^2+r^4(4R+r)^3$$
$$=r(4R^2-8Rr+6r^2)s^4-\left(6R^5-3r^5\right)s^2+r^4(4R+r)^3$$
$$\overset{\text{Gerretsen}}{\le}\left(r(4R^2-8Rr+6r^2)(4R^2+4Rr+3r^2)-\left(6R^5-3r^5\right)\right)s^2+r^4(4R+r)^3$$
$$\overset{?}{\le}0\Leftrightarrow\left(6R^5-16R^4r+16R^3r^2-4R^2r^3-21r^5\right)s^2\ \boxed{\underset{(**)}{\overset{?}{\ge}}}\ r^4(4R+r)^3$$

$$\text{Now, }6R^5-16R^4r+16R^3r^2-4R^2r^3-21r^5=$$
$$(R-2r)(4R^4+2R^3(R-2r)+8R^2r^2+12Rr^3+24r^4)+27r^5\overset{\text{Euler}}{\ge}27r^5>0$$
$$\therefore\text{ LHS of }(**)\overset{\text{Gerretsen}}{\ge}\left(6R^5-16R^4r+16R^3r^2-4R^2r^3-21r^5\right)(16Rr-5r^2)\overset{?}{\ge}$$
$$r^4(4R+r)^3\Leftrightarrow 48t^6-143t^5+168t^4-104t^3-14t^2-174t+52\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t-2)\left(24t^5+24t^4(t-2)+t^4+74t^3+44t^2+61t+13(t-2)\right)\overset{?}{\ge}0\to\text{true}$$
$$\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true }\therefore\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}\le3\left(\frac{R}{2r}\right)^3$$

$$\text{Again, }\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}=$$
$$\frac{(s^2+4Rr+r^2)^3-3(4Rrs)\left(2s(s^2+2Rr+r^2)\right)}{16R^2r^2s^2}\overset{?}{\ge}3$$
$$\Leftrightarrow s^6-(12Rr-3r^2)s^4-r^2s^2(48R^2-3r^2)+r^3(4R+r)^3\ \boxed{\underset{(***)}{\overset{?}{\ge}}}\ 0$$

$$\text{Also, LHS of }(***)\overset{\text{Gerretsen}}{\ge}(4Rr-2r^2)s^4-r^2s^2(48R^2-3r^2)+r^3(4R+r)^3$$

$$\overset{\text{Gerretsen}}{\geq}\left((4Rr-2r^2)(16Rr-5r^2)-r^2(48R^2-3r^2)\right)s^2+r^3(4R+r)^3\overset{?}{\geq}0$$

$$\Leftrightarrow (16R^2-52Rr+13r^2)s^2+r(4R+r)^3\boxed{\underset{(****)}{\overset{?}{\geq}}}0$$

$\boxed{Case\ 1}$ $16R^2-52Rr+13r^2\geq 0$ and then : LHS of $(****)\geq r(4R+r)^3>0$
$\Rightarrow (****)$ is true (strict inequality)

$\boxed{Case\ 2}$ $16R^2-52Rr+13r^2<0$ and then : LHS of $(****)\overset{\text{Gerretsen}}{\geq}$

$$(16R^2-52Rr+13r^2)(4R^2+4Rr+3r^2)+r(4R+r)^3\overset{?}{\geq}0$$

$$\Leftrightarrow 16t^4-20t^3-15t^2-23t+10\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)(16t^3+12t^2+3(t-2)+6t+1)\overset{?}{\geq}0\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow (****)$ is true and so, combining both cases, $(****)\Rightarrow(***)$ is true $\forall\,\Delta\,ABC$

$$\therefore \frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}\geq 3 \text{ and so, } 3\leq\frac{h_a}{a\sin A}+\frac{h_b}{b\sin A}+\frac{h_c}{c\sin C}\leq 3\left(\frac{R}{2r}\right)^3$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)