ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationship holds :**

$$-1+19\left(\frac{r}{R}\right)^4\le\sum_{cyc}\cos^2 B\cos^2 C\le\frac{137}{32}-\frac{131}{2}\left(\frac{r}{R}\right)^4$$

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$$\sum_{cyc}\cos^2 B\cos^2 C=\sum_{cyc}\left(\left(1-\frac{b^2}{4R^2}\right)\left(1-\frac{c^2}{4R^2}\right)\right)=$$

$$=3-\sum_{cyc}\frac{b^2+c^2}{4R^2}+\sum_{cyc}\frac{b^2c^2}{16R^4}\overset{\text{Goldstone}}{\le}3-\frac{s^2-4Rr-r^2}{R^2}+\frac{4R^2s^2}{16R^4}=$$

$$=\frac{12R^2-4s^2+16Rr+4r^2+s^2}{4R^2}\le$$

$$\overset{\text{Gerretsen}}{\le}\frac{12R^2+16Rr+4r^2-3(16Rr-5r^2)}{4R^2}\overset{?}{\le}\frac{137R^4-16.131r^4}{32R^4}$$

$$\Leftrightarrow 41t^4+256t^3-152t^2-2096\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(41t^3+338t^2+524t+1048)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\therefore\sum_{cyc}\cos^2 B\cos^2 C\le\frac{137}{32}-\frac{131}{2}\left(\frac{r}{R}\right)^4$$

$$\text{Now,}\left(\sum_{cyc}ab\right)^2-24Rrs^2=$$

$$=s^4+(4Rr+r^2)^2+2(4Rr+r^2)s^2-24Rrs^2\overset{\text{Gerretsen}}{\ge}$$

$$\ge(16Rr-5r^2)s^2+2(4Rr+r^2)s^2-24Rrs^2+(4Rr+r^2)^2=r^2((4R+r)^2-3s^2)$$

$$\overset{\text{Doucet or Trucht}}{\ge}0\Rightarrow\sum_{cyc}b^2c^2\ge\frac{1}{3}\left(\sum_{cyc}ab\right)^2\ge\frac{24Rrs^2}{3}\Rightarrow\sum_{cyc}b^2c^2\ge 8Rrs^2\to(1)$$

$$\therefore\sum_{cyc}\cos^2 B\cos^2 C=3-\frac{s^2-4Rr-r^2}{R^2}+\sum_{cyc}\frac{b^2c^2}{16R^4}\overset{\text{via (1)}}{\ge}$$

$$\ge 3-\frac{s^2-4Rr-r^2}{R^2}+\frac{8Rrs^2}{16R^4}=\frac{6R^4-2R^2(s^2-4Rr-r^2)+Rrs^2}{2R^4}=$$

$$=\frac{6R^4+2R^2(4Rr+r^2)-(2R^2-Rr)s^2}{2R^4}=$$

$$\overset{\text{Gerretsen}}{\ge}\frac{6R^4+2R^2(4Rr+r^2)-(2R^2-Rr)(4R^2+4Rr+3r^2)}{2R^4}\overset{?}{\ge}-1+19\left(\frac{r}{R}\right)^4$$

$$=\frac{19r^4-R^4}{R^4}\Leftrightarrow 4t^3+3t-38\overset{?}{\geq}0\Leftrightarrow(t-2)(4t^2+8t+19)\overset{?}{\geq}0\rightarrow \text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\therefore\sum_{\text{cyc}}\cos^2 B\cos^2 C\geq -1+19\left(\frac{r}{R}\right)^4 \text{ and so, } -1+19\left(\frac{r}{R}\right)^4\leq$$

$$\sum_{\text{cyc}}\cos^2 B\cos^2 C\leq\frac{137}{32}-\frac{131}{2}\left(\frac{r}{R}\right)^4\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$