ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$p^2+\frac{\lambda}{\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}}\ge(27+3\lambda)r^2, 0\le\lambda\le\frac{9}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

For own convenience, $p\equiv s$

$$\text{Firstly, }\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\ge\frac{1}{3}\left(\sum_{cyc}\frac{1}{r_a}\right)^2=\frac{1}{3r^2}\Rightarrow\sum_{cyc}\frac{1}{r_a^2}\ge\frac{1}{3r^2}$$

$$\Rightarrow 3r^2-\frac{1}{\sum_{cyc}\frac{1}{r_a^2}}\ge0\to(1)$$

$$s^2+\frac{\lambda}{\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}}\ge(27+3\lambda)r^2\Leftrightarrow s^2-27r^2\ge\lambda\left(3r^2-\frac{1}{\sum_{cyc}\frac{1}{r_a^2}}\right)$$

to prove which it suffices to prove :

$$s^2-27r^2\ge\frac{9}{2}\left(3r^2-\frac{1}{\sum_{cyc}\frac{1}{r_a^2}}\right)\left(\text{via }(1)\text{ and }\because 0\le\lambda\le\frac{9}{2}\right)$$

$$\Leftrightarrow s^2-27r^2\ge\frac{9}{2}\left(3r^2-\frac{r^2s^4}{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}\right)$$

$$=\frac{9}{2}\left(3r^2-\frac{r^2s^4}{s^4-2rs^2(4R+r)}\right)\Leftrightarrow s^2-27r^2\ge\frac{9r^2}{2}\left(\frac{2s^2-24Rr-6r^2}{s^2-8Rr-2r^2}\right)$$

$$\Leftrightarrow(s^2-8Rr-2r^2)(s^2-27r^2)\overset{(*)}{\ge}9r^2(s^2-12Rr-3r^2)$$

$$s^2-27r^2\overset{?}{\ge}s^2-12Rr-3r^2\Leftrightarrow12Rr\overset{?}{\ge}24r^2\to\text{true via Euler}$$

$$\therefore s^2-27r^2\ge s^2-12Rr-3r^2\to(2)$$

$$\text{Also, }s^2-8Rr-2r^2\overset{?}{\ge}9r^2\Leftrightarrow s^2-16Rr+5r^2+8r(R-2r)\overset{?}{\ge}0\to\text{true}$$

$$\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0\text{ and }8r(R-2r)\overset{\text{Euler}}{\ge}0\therefore s^2-8Rr-2r^2\ge9r^2\to(3)$$

$$\therefore(2)\bullet(3)\Rightarrow(*)\text{ is true }\therefore$$

$$p^2+\frac{\lambda}{\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}}\ge(27+3\lambda)r^2\ \forall\,\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$