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In ΔABC holds:

$$\sum_{cyc}\frac{h_a(2h_a-3r)}{{h_a}^2+18r^2}\geq 1$$

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$$\sum_{cyc}\frac{h_a(2h_a-3r)}{{h_a}^2+18r^2}=\sum_{cyc}\left(2-\frac{3r(h_a+12r)}{({h_a}^2+9r^2)+9r^2}\right)\overset{AM-GM}{\geq}\sum_{cyc}\left(2-\frac{3r(h_a+12r)}{6rh_a+9r^2}\right)=$$

$$=\sum_{cyc}\left(2-\frac{h_a+12r}{2h_a+3r}\right)=\sum_{cyc}\left(\frac{3}{2}-\frac{21r}{2(2h_a+3r)}\right)\overset{CBS}{\geq}\sum_{cyc}\left[\frac{3}{2}-\frac{21r}{2.9}\left(\frac{2}{h_a}+\frac{1}{3r}\right)\right]=$$

$$=\sum_{cyc}\left(\frac{10}{9}-\frac{7r}{3h_a}\right)=\frac{10}{3}-\frac{7r}{3}.\sum_{cyc}\frac{1}{h_a}=\frac{10}{3}-\frac{7r}{3}.\frac{1}{r}=1.$$

Equality holds iff ΔABC is equilateral.