ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$3r \le \frac{\sum r_a \sin A}{\sum \sin A} \le 2R - r$$

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Solution by Tapas Das-India

$$\sum r_a \sin A = \sum \frac{F}{s-a}\frac{a}{2R} = \frac{F}{2R}\sum \frac{a}{s-a} = \frac{F}{2R}\cdot \frac{\sum a(s-b)(s-c)}{(s-a)(s-b)(s-c)} =$$

$$= \frac{F}{2R}\cdot \frac{\left(\sum (as^2 - s(ab+ac) + abc)\right)}{sr^2} =$$

$$= \frac{F}{2R}\cdot \frac{s^2(a+b+c) - 2s(ab+bc+ca) + 3abc}{sr^2} =$$

$$= \frac{2s^3 - 2s(s^2+r^2+4Rr) + 12Rrs}{sr^2}\cdot \frac{F}{2R} = \frac{2sr(2R-r)}{sr^2}\frac{rs}{2R} = (2R-r)\frac{s}{R}\ (1)$$

$$\frac{\sum r_a \sin A}{\sum \sin A} \overset{(1)}{=} (2R-r)\frac{s}{R}\frac{1}{\frac{s}{R}} = 2R - r \overset{Euler}{\ge} 4r - r = 3r$$

$$Equality\ holds\ \ for\ a = b = c$$