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In ΔABC the following relationship holds:

$$\sum\left(\frac{AI}{b+c}\right)^2 \ge \frac{r^2}{R^2}$$

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Solution by Tapas Das-India

$$AI=\frac{r}{\sin\frac{A}{2}},\ BI=\frac{r}{\sin\frac{B}{2}},\ CI=\frac{r}{\sin\frac{C}{2}}\ \ and$$

$$AI\cdot BI\cdot CI=\frac{r^3}{\prod\sin\frac{A}{2}}=r^3\frac{4R}{r}\overset{Euler}{\ge} r^2.4.2r=8r^3=(2r)^3\ \ (1)$$

$$\sqrt[3]{(a+b)(b+c)(c+a)}\overset{AM-GM}{\le}\frac{2(a+b+c)}{3}=\frac{4s}{3}\overset{Mitrinovic}{\le}\frac{4}{3}\cdot\frac{3\sqrt{3}R}{2}=2\sqrt{3}R\ \ (2)$$

$$\sum\left(\frac{AI}{b+c}\right)^2\overset{AM-GM}{\ge}3\left(\sqrt[3]{\frac{AI\cdot BI\cdot CI}{(a+b)(b+c)(c+a)}}\right)^2\overset{(1)\&(2)}{\ge}3.\frac{\left((2r)^3\right)^{\frac{2}{3}}}{\left(2\sqrt{3}R\right)^2}=\frac{12r^2}{12R^2}=\frac{r^2}{R^2}$$

Equality holds for an equilateral triangle