ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$9r.\left(\frac{2r}{R}\right)^{\frac{1}{3}} \le \sum_{cyc}\left((b+c-a).\cos\frac{A}{2}\right) \le \frac{9R}{2}$$

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$$\left(\sum_{cyc}\left((s-a).\cos\frac{A}{2}\right)\right)^2 =$$

$$= \sum_{cyc}\left((s-a)^2\left(1-\sin^2\frac{A}{2}\right)\right) + 2(s-b)(s-c).\cos\frac{B}{2}.\cos\frac{C}{2}$$

$$= \sum_{cyc}(s^2-2sa+a^2) - \sum_{cyc}\left((s-a)^2.\frac{(s-b)(s-c)}{bc}\right)$$

$$+\frac{2s}{4R}.\sum_{cyc}\left((s-b)(s-c).\sec\frac{A}{2}\right)$$

$$\overset{CBS}{\le} \left(3s^2-4s^2+2(s^2-4Rr-r^2)\right) - \frac{r^2s}{4Rrs}.\sum_{cyc}a(s-a) +$$

$$\frac{s}{2R}.\sqrt{\sum_{cyc}\left((s-b)(s-c)\right)^2}.\sqrt{\frac{s^2+(4R+r)^2}{s^2}}\left(\because \sum_{cyc}\sec^2\frac{A}{2} = \frac{s^2+(4R+r)^2}{s^2}\right)$$

$$= s^2-8Rr-2r^2-\frac{r^2s}{4Rrs}.\left(s(2s)-2(s^2-4Rr-r^2)\right)+$$

$$\frac{1}{2R}.\sqrt{(4Rr+r^2)^2-2r^2s^2}.\sqrt{s^2+(4R+r)^2}$$

$$\left(\because \sum_{cyc}\left((s-b)(s-c)\right)^2 = \left(\sum_{cyc}(s-b)(s-c)\right)^2 - 2s\left(\prod_{cyc}(s-a)\right)\right)$$

$$= \frac{2Rs^2-r(4R+r)^2}{2R} + \frac{1}{2R}.\sqrt{(4Rr+r^2)^2-2r^2s^2}.\sqrt{s^2+(4R+r)^2} \overset{?}{\le} \frac{81R^2}{16}$$

$$\Leftrightarrow \frac{81R^3+8r(4R+r)^2-16Rs^2}{16R} \boxed{\overset{?}{\underset{(*)}{\ge}}} \frac{1}{2R}.\sqrt{(4Rr+r^2)^2-2r^2s^2}.\sqrt{s^2+(4R+r)^2}$$

$$\text{Now, } 81R^3+8r(4R+r)^2-16Rs^2 \overset{Gerretsen}{\ge}$$

$$81R^3+8r(4R+r)^2-16R(4R^2+4Rr+3r^2) = 17R^3+64R^2r+16Rr^2+8r^3 > 0$$

$$\therefore (*) \Leftrightarrow (81R^3+8r(4R+r)^2-16Rs^2)^2$$
$$\geq 64((4Rr+r^2)^2-2r^2s^2)(s^2+(4R+r)^2)$$
$$\Leftrightarrow (256R^2+128r^2)s^4-(2592R^4+4096R^3r+1024R^2r^2-256Rr^3-64r^4)s^2$$
$$+6561R^6+20736R^5r+10368R^4r^2+1296R^3r^3 \boxed{\overset{(**)}{\geq}} 0 \textit{ and } \because$$
$$(256R^2+128r^2)(s^2-4R^2-4Rr-3r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$
$$\text{it suffices to prove : LHS of } (**) \geq (256R^2+128r^2)(s^2-4R^2-4Rr-3r^2)^2 \Leftrightarrow$$
$$2465R^6+12544R^5r-1920R^4r^2-8944R^3r^3-7424R^2r^4-3072Rr^5-1152r^6$$
$$\boxed{\overset{(***)}{\geq}} (544R^4+2048R^3r-1536R^2r^2-1280Rr^3-832r^4)s^2$$
$$\text{Now, } 544R^4+2048R^3r-1536R^2r^2-1280Rr^3-832r^4=$$
$$(R-2r)(544R^3+3136R^2r+4736Rr^2+8192r^3)+15552r^4 \overset{\text{Euler}}{\geq} 15552r^4>0$$
$$\therefore \text{RHS of } (***) \overset{\text{Gerretsen}}{\leq} \begin{pmatrix} 544R^4+2048R^3r-1536R^2r^2 \\ -1280Rr^3-832r^4 \end{pmatrix}(4R^2+4Rr+3r^2)$$
$$\overset{?}{\leq} \text{LHS of } (***)$$
$$\Leftrightarrow 289t^6+2176t^5-5600t^4-3824t^3+5632t^2+4096t+1344 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(289t^4+3332t^3+6572t^2+9136t+15888)+31104\right) \overset{?}{\geq} 0$$
$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \left(\sum_{\text{cyc}}\left((s-a).\cos\frac{A}{2}\right)\right)^2 \leq \frac{81R^2}{16}$$
$$\Rightarrow \sum_{\text{cyc}}\left((s-a).\cos\frac{A}{2}\right) \leq \frac{9R}{4} \Rightarrow \sum_{\text{cyc}}\left((b+c-a).\cos\frac{A}{2}\right) \leq \frac{9R}{2}$$
$$\text{Again, } \sum_{\text{cyc}}\left((b+c-a).\cos\frac{A}{2}\right) \overset{A-G}{\geq} 6.\sqrt[3]{\left(\prod_{\text{cyc}}(s-a)\right).\left(\prod_{\text{cyc}}\cos\frac{A}{2}\right)} = 6.\sqrt[3]{\frac{r^2(2s^2)}{8R}}$$
$$\overset{\text{Gerretsen + Euler}}{\geq} 3.\sqrt[3]{\frac{r^2(27Rr)}{R}} = 9r \overset{\text{Euler}}{\geq} 9r.\left(\frac{2r}{R}\right)^{\frac{1}{3}} \therefore \sum_{\text{cyc}}\left((b+c-a).\cos\frac{A}{2}\right)$$
$$\geq 9r.\left(\frac{2r}{R}\right)^{\frac{1}{3}} \textit{ and so, } 9r.\left(\frac{2r}{R}\right)^{\frac{1}{3}} \leq \sum_{\text{cyc}}\left((b+c-a).\cos\frac{A}{2}\right) \leq \frac{9R}{2}$$
$$\forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$