ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$18r \le \sum (b+c-a)\cot\frac{A}{2} \le \frac{2(2R-r)^2}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum (b+c-a)\cot\frac{A}{2} = \sum 2(s-a)\cot\frac{A}{2} = 2\sum (s-a)\frac{s}{r_a} =$$

$$= 2s^2\sum\frac{1}{r_a} - 2s\sum\frac{a(s-a)}{rs} = \frac{2s^2}{r} - \frac{2}{r}\left(s\sum a - \sum a^2\right) =$$

$$= \frac{2s^2}{r} - \frac{2}{r}(2s^2 - 2(s^2 - r^2 - 4Rr) = \frac{2}{r}(s^2 - 8Rr - 2r^2) \overset{GERRETSEN}{\lessapprox}$$

$$\le \frac{2}{r}(4R^2 + 4Rr + 3r^2 - 8Rr - 2r^2) = \frac{2}{r}(4R^2 - 4Rr + r^2) = \frac{2(2R-r)^2}{r}$$

$$\sum (b+c-a)\cot\frac{A}{2} = \sum 2(s-a)\cot\frac{A}{2} = 2\sum (s-a)\frac{s}{r_a} =$$

$$= 2s\sum\frac{(s-a)}{r_a} \overset{AM-GM}{\ge} 6s(\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{r_ar_br_c}} = 6s\sqrt[3]{\frac{sr^2}{s^2r}} = 6s\sqrt[3]{\frac{r}{s}} =$$

$$= 6s\sqrt[3]{\frac{rs^2}{s^3}} \overset{Mitrinovic}{\ge} 6s\sqrt[3]{\frac{27r^3}{s^3}} = 18r$$

Equality holds for an equilateral triangle