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In ΔABC the following relationship holds:

$$6r \le \sum (b+c-a)\tan\frac{A}{2} \le 3R$$

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Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c\ then\ \tan\frac{A}{2} \ge \tan\frac{B}{2} \ge \tan\frac{C}{2}$$

$$\sum (b+c-a)\tan\frac{A}{2} = 2\sum (s-a)\tan\frac{A}{2} \overset{Chebyshev}{\le} 2\cdot\frac{1}{3}\sum (s-a)\sum \tan\frac{A}{2} =$$

$$= \frac{2}{3}s.\frac{4R+r}{s} \overset{Euler}{\le} \frac{2}{3}\frac{9R}{2} = 3R$$

$$\sum (b+c-a)\tan\frac{A}{2} = 2\sum (s-a)\tan\frac{A}{2} \overset{AM-GM}{\ge}$$

$$\ge 6\left(\prod (s-a)\tan\frac{A}{2}\right)^{\frac{1}{3}} = 6\left(\frac{sr^2r}{s}\right)^{\frac{1}{3}} = 6r$$

Equality holds for $a=b=c$.