# ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationship holds :**

$$\frac{8R}{r} \le \sum_{cyc} \frac{(b+c)^2}{r_a^2} \le 16\left(\frac{R^2}{r^2} - 3\right)$$

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WLOG we may assume $a \ge b \ge c$ and then :

$$(b+c)^2 \le (c+a)^2 \le (a+b)^2 \text{ and } \frac{1}{r_a^2} \le \frac{1}{r_b^2} \le \frac{1}{r_c^2} \therefore \text{ via Chebyshev,}$$

$$\sum_{cyc} \frac{(b+c)^2}{r_a^2} \ge \left(\sum_{cyc} (b+c)^2\right)\left(\sum_{cyc} \frac{1}{r_a^2}\right)$$

$$\overset{A-G}{\ge} 4\left(\sum_{cyc} ab\right) \cdot \frac{\left(\sum_{cyc} r_b r_c\right)^2 - 2r_a r_b r_c \sum_{cyc} r_a}{r^2 s^4}$$

$$= 4(s^2 + 4Rr + r^2) \cdot \frac{s^4 - 2rs^2(4R+r)}{r^2 s^4} \overset{?}{\ge} \frac{8R}{r}$$

$$\Leftrightarrow s^4 - (10Rr + r^2)s^2 - 2r^2(4R+r)^2 \underset{(*)}{\overset{?}{\ge}} 0$$

$$\text{Now, LHS of } (*) \overset{Gerretsen}{\ge} (6Rr - 6r^2)s^2 - 2r^2(4R+r)^2 \overset{Gerretsen}{\ge}$$

$$(6Rr - 6r^2)(16Rr - 5r^2) - 2r^2(4R+r)^2 \overset{?}{\ge} 0 \Leftrightarrow 32R^2 - 71Rr + 14r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R - 2r)(32R - 7r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{(b+c)^2}{r_a^2} \ge \frac{8R}{r}$$

$$\text{Again, } \sum_{cyc} \left((b+c)^2 (s-a)^2\right) = \sum_{cyc} \left((s + s - a)^2 (s-a)^2\right)$$

$$= \sum_{cyc} \left(\left(s^2 + (s-a)^2 + 2s(s-a)\right)(s-a)^2\right)$$

$$= s^2 \sum_{cyc} (s^2 - 2sa + a^2) + 2s \sum_{cyc} (s^3 - 3s^2 a + 3sa^2 - a^3)$$

$$+ \sum_{cyc} (s^4 - 4s^3 a + 6s^2 a^2 - 4sa^3 + a^4)$$

$$= s^2 . 3s^2 - 2s^3 . 2s + s^2 \sum_{cyc} a^2 + 2s . 3s^3 - 6s^3 . 2s + 6s^2 \sum_{cyc} a^2 - 2s \sum_{cyc} a^3 + 3s^4$$

$$-4s^3 . 2s + 6s^2 \sum_{cyc} a^2 - 4s \sum_{cyc} a^3 + 2 \sum_{cyc} a^2 b^2 - 16r^2 s^2$$

$$= -12s^4 + 26s^2(s^2 - 4Rr - r^2) - 12s^2(s^2 - 6Rr - 3r^2)$$

$$+2\left((s^2 + 4Rr + r^2)^2 - 16Rrs^2\right) - 16r^2 s^2$$

$$= 2\left(2s^4 - (24Rr + r^2)s^2 + r^2(4R+r)^2\right)$$

$$\Rightarrow \sum_{cyc} \frac{(b+c)^2}{r_a^2} = \frac{2\left(2s^4 - \left(24Rr + r^2\right)s^2 + r^2(4R+r)^2\right)}{r^2 s^2} \overset{?}{\leq} 16\left(\frac{R^2}{r^2} - 3\right)$$

$$\Leftrightarrow 2s^4 + r^2(4R+r)^2 \underset{(**)}{\underbrace{\overset{?}{\leq}}} \left(8R^2 + 24Rr - 23r^2\right)s^2$$

Now, LHS of $(**) \overset{\text{Gerretsen}}{\leq} \left(8R^2 + 8Rr + 6r^2\right)s^2 + r^2(4R+r)^2 \overset{?}{\leq}$

$$\left(8R^2 + 24Rr - 23r^2\right)s^2 \Leftrightarrow \left(16Rr - 29r^2\right)s^2 \underset{(***)}{\underbrace{\overset{?}{\geq}}} r^2(4R+r)^2$$

Again, LHS of $(***) \overset{\text{Gerretsen}}{\geq} \left(16Rr - 29r^2\right)\left(16Rr - 5r^2\right) \overset{?}{\geq} r^2(4R+r)^2$

$$\Leftrightarrow 240R^2 - 552Rr + 144r^2 \overset{?}{\geq} 0 \Leftrightarrow 24(R-2r)(10R-3r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (***) \Rightarrow (**) \text{ is true} \therefore \sum_{cyc} \frac{(b+c)^2}{r_a^2} \leq 16\left(\frac{R^2}{r^2} - 3\right) \therefore \frac{8R}{r} \leq \sum_{cyc} \frac{(b+c)^2}{r_a^2}$$

$$\leq 16\left(\frac{R^2}{r^2} - 3\right) \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$