ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the $following$ $relationship$ holds :

$$12r^2 \le \sum_{cyc} \frac{a^4}{4s(s-a)} \le \frac{4(R^3-5r^3)}{r}$$

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$$\sum_{cyc} \frac{a^4}{s(s-a)} = \sum_{cyc} \frac{\left(s-(s-a)\right)^4}{s(s-a)} =$$

$$= \sum_{cyc} \frac{s^4+(s-a)^4-4s^3(s-a)-4s(s-a)^3+6s^2(s-a)^2}{s(s-a)}$$

$$= \frac{s^3(4Rr+r^2)}{r^2 s} + \frac{1}{s}.\left(\left(\sum_{cyc}(s-a)\right)^3 - 3abc\right) - 12s^2$$

$$-4\left(\left(\sum_{cyc}(s-a)\right)^2 - 2\sum_{cyc}(s-b)(s-c)\right) + 6s.\sum_{cyc}(s-a)$$

$$= \frac{s^2(4R+r)}{r} + s^2 - 12Rr - 12s^2 - 4s^2 + 8(4Rr+r^2) + 6s^2$$

$$= \frac{s^2(4R+r)}{r} - 9s^2 + 20Rr + 8r^2$$

$$\therefore \sum_{cyc} \frac{a^4}{s(s-a)} = \frac{s^2(4R-8r)+r^2(20R+8r)}{r} \to (1)$$

Via (1) and Gerretsen, $\sum_{cyc} \frac{a^4}{s(s-a)} \le \frac{(4R-8r)(4R^2+4Rr+3r^2)+r^2(20R+8r)}{r}$

$$\overset{?}{\le} \frac{16(R^3-5r^3)}{r} \Leftrightarrow 16r(R^2-4r^2) \overset{?}{\ge} 0 \to \text{true via Euler}$$

$$\therefore \sum_{cyc} \frac{a^4}{4s(s-a)} \le \frac{4(R^3-5r^3)}{r}$$

Via (1) and Gerretsen, $\sum_{cyc} \frac{a^4}{s(s-a)} \ge \frac{(4R-8r)(16Rr-5r^2)+r^2(20R+8r)}{r}$

$\overset{?}{\ge} 48r^2 \Leftrightarrow 64R(R-2r) \overset{?}{\ge} 0 \to$ true via Euler $\therefore \sum_{cyc} \frac{a^4}{4s(s-a)} \ge 12r^2$ and so,

$$12r^2 \le \sum_{cyc} \frac{a^4}{4s(s-a)} \le \frac{4(R^3-5r^3)}{r} \ \forall \Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$