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In any Δ ABC, the following relationship holds :

$$\frac{18R^2}{s} \le \sum_{cyc} \frac{a^3}{s(s-a)} \le \frac{8(2R^3-7r^3)}{sr}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a^3}{s(s-a)} = \sum_{cyc} \frac{\left(s-(s-a)\right)^3}{s(s-a)} =$$

$$= \sum_{cyc} \frac{s^3-(s-a)^3-3s^2(s-a)+3s(s-a)^2}{s(s-a)} =$$

$$= \frac{s^2(4Rr+r^2)}{r^2s} - \frac{1}{s}.\left(\left(\sum_{cyc}(s-a)\right)^2 - 2\sum_{cyc}(s-b)(s-c)\right) - 9s + 3\sum_{cyc}(s-a)$$

$$= \frac{s(4R+r)}{r} - s + \frac{2(4Rr+r^2)}{s} - 6s$$

$$\therefore \sum_{cyc} \frac{a^3}{s(s-a)} = \frac{s^2(4R-6r)+r^2(8R+2r)}{rs} \to (1)$$

Via (1) and Gerretsen, $\sum_{cyc} \frac{a^3}{s(s-a)} \le \frac{(4R-6r)(4R^2+4Rr+3r^2)+r^2(8R+2r)}{rs}$

$$\overset{?}{\le} \frac{8(2R^3-7r^3)}{sr} \Leftrightarrow 4r(2R^2+Rr-10r^2) \overset{?}{\ge} 0 \Leftrightarrow 4r(R-2r)(2R+5r) \overset{?}{\ge} 0$$

$\to$ true via Euler $\therefore \sum_{cyc} \frac{a^3}{s(s-a)} \le \frac{8(2R^3-7r^3)}{sr}$

Via (1) and Gerretsen, $\sum_{cyc} \frac{a^3}{s(s-a)} \ge \frac{(4R-6r)(16Rr-5r^2)+r^2(8R+2r)}{rs}$

$$\overset{?}{\ge} \frac{18R^2}{s} \Leftrightarrow 23R^2-54Rr+16r^2 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(23R-8r) \overset{?}{\ge} 0 \to \text{true via Euler}$$

$$\therefore \sum_{cyc} \frac{a^3}{s(s-a)} \ge \frac{18R^2}{s} \text{ and so, } \frac{18R^2}{s} \le \sum_{cyc} \frac{a^3}{s(s-a)} \le \frac{8(2R^3-7r^3)}{sr}$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)