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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{3r}{r_a}+2}\ge\frac{6}{\frac{3r(4R+r)}{s^2}+5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\frac{3r}{r_a}+2}=\sum_{cyc}\frac{1}{\frac{3r(s-a)}{rs}+2}=s.\sum_{cyc}\frac{1}{5s-3a}$$

$$=s.\frac{\sum_{cyc}\left((5s-3a)(5s-3b)\right)}{(5s-3a)(5s-3b)(5s-3c)}=\frac{3s\left(25s^2+3\sum_{cyc}ab-10s\sum_{cyc}a\right)}{s\left(125s^2+45\sum_{cyc}ab-75s\sum_{cyc}a-108Rr\right)}$$

$$\ge\frac{6s^2}{3r(4R+r)+5s^2}\Leftrightarrow(4R-17r)s^2+3r(4R+r)^2\overset{(*)}{\ge}0$$

$$\text{Now, }(4R-17r)s^2+3r(4R+r)^2=(4R-8r)s^2-9rs^2+3r(4R+r)^2$$

$$\overset{\text{Gerretsen}}{\ge}(4R-8r)(16Rr-5r^2)-9r(4R^2+4Rr+3r^2)+3r(4R+r)^2\overset{?}{\ge}0$$

$$\Leftrightarrow 19R^2-40Rr+4r^2\overset{?}{\ge}0\Leftrightarrow(19R-2r)(R-2r)\overset{?}{\ge}0\to\text{true via Euler}\Rightarrow$$

$$(*)\text{ is true}\therefore\sum_{cyc}\frac{1}{\frac{3r}{r_a}+2}\ge\frac{6}{\frac{3r(4R+r)}{s^2}+5}\ \forall\,\Delta\text{ ABC},''=''\ \text{iff }\Delta\text{ ABC is equilateral (QED)}$$