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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{3r}{h_a}+2}\ge\frac{6}{\frac{3(R+r)^2}{s^2}+5}$$

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$$\sum_{cyc}\frac{1}{\frac{3r}{h_a}+2}=\sum_{cyc}\frac{1}{\frac{3ra}{2rs}+2}=2s.\sum_{cyc}\frac{1}{3a+4s}$$

$$=2s.\frac{\sum_{cyc}\big((4s+3a)(4s+3b)\big)}{(4s+3a)(4s+3b)(4s+3c)}=\frac{6s\left(16s^2+3\sum_{cyc}ab+8s\sum_{cyc}a\right)}{s\left(64s^2+36\sum_{cyc}ab+48s\sum_{cyc}a+108Rr\right)}$$

$$\ge\frac{6s^2}{3(R+r)^2+5s^2}$$

$$\Leftrightarrow -7s^4+(35R^2+6Rr+28r^2)s^2+r(12R^3+27R^2r+18Rr^2+3r^3)\overset{(*)}{\ge}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}\Big(-7(4R^2+4Rr+3r^2)+(35R^2+6Rr+28r^2)\Big)s^2$$

$$+r(12R^3+27R^2r+18Rr^2+3r^3)\overset{?}{\ge}0$$

$$\Leftrightarrow (7R^2-22Rr+7r^2)s^2+r(12R^3+27R^2r+18Rr^2+3r^3)\underset{(**)}{\overset{?}{\ge}}0$$

$\boxed{\text{Case 1}}$ $7R^2-22Rr+7r^2\ge 0$ and then : LHS of $(**)\ge$ $r(12R^3+27R^2r+18Rr^2+3r^3)>0\Rightarrow(**)$ is true

$\boxed{\text{Case 2}}$ $7R^2-22Rr+7r^2<0$ and then : LHS of $(**)\overset{\text{Gerretsen}}{\ge}$

$$(7R^2-22Rr+7r^2)(4R^2+4Rr+3r^2)+r(12R^3+27R^2r+18Rr^2+3r^3)\overset{?}{\ge}0$$

$$\Leftrightarrow 7t^4-12t^3-3t^2-5t+6\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(7t^3+2(t^2-4)+t+5)\overset{?}{\ge}0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)$ is true $\therefore$ combining both cases, $(**)\Rightarrow(*)$ is true

$\forall\,\Delta$ ABC $\therefore\displaystyle\sum_{cyc}\frac{1}{\frac{3r}{h_a}+2}\ge\frac{6}{\frac{3(R+r)^2}{s^2}+5}$ $\forall\,\Delta$ ABC, "=" iff Δ ABC is equilateral (QED)