ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \cot A + \lambda \prod \cot\frac{A}{2} \ge \frac{1}{2}(3\lambda+1)\sum \csc A, \lambda \ge 0$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

It is known: In ΔABC:

$$\sum \tan A \tan B = \frac{s^2-4Rr-r^2}{s^2-(2R+r)^2}, \prod \tan A = \frac{2sr}{s^2-(2R+r)^2},$$

$$\prod \sin A = \frac{sr}{2R^2}, \sum \sin A \sin B = \frac{s^2+4Rr+r^2}{4R^2}$$

Using above result we get:

$$\sum \cot A = \frac{s^2-4Rr-r^2}{2sr}, \sum \csc A = \frac{s^2+4Rr+r^2}{2sr},$$

$$\prod \cot\frac{A}{2} = \frac{s}{r}$$

We need to show:

$$\sum \cot A + \lambda \prod \cot\frac{A}{2} \ge \frac{1}{2}(3\lambda+1)\sum \csc A$$

$$\frac{s^2-4Rr-r^2}{2sr} + \lambda\frac{s}{r} \ge \frac{1}{2}(3\lambda+1)\frac{s^2+4Rr+r^2}{2sr}$$

$$2s^2-8Rr-2r^2+4\lambda s^2 \ge (3\lambda+1)(s^2+r^2+4Rr)$$

$$2s^2-8Rr-2r^2+4\lambda s^2 \ge 3\lambda s^2+3\lambda r(4R+r)+s^2+r^2+4Rr$$

$$s^2-12Rr-3r^2+4\lambda s^2 \ge 3\lambda s^2+3\lambda r(4R+r)$$

$$s^2-12Rr-3r^2+4\lambda s^2 \overset{s^2\ge 3r(4R+r)}{\ge} 3\lambda s^2+\lambda s^2$$

$$s^2-12Rr-3r^2 \ge 0$$
$$16Rr-5r^2-12Rr-3r^2 \ge 0 \ (Gerretsen)$$
$$or\ 4Rr-8r^2 \ge 0\ or\ R \ge 2r\ true\ Euler$$
$$Equality\ holds\ for\ A=B=C$$