ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\prod\left(\frac{8}{r_a^2}+\frac{1}{r_ar_b}\right)\le\frac{1}{r^6}$$

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$$Let\ \frac{r}{r_a}=x,\frac{r}{r_b}=y,\frac{r}{r_c}=z\ \ then:$$

$$x+y+z=r\sum\frac{1}{r_a}=1\ (1)\ and\ xyz\overset{AM-GM}{\le}\left(\frac{x+y+z}{3}\right)^3=\frac{1}{27}\ (2)$$

$$(8x+y)(8y+z)(8z+x)\overset{AM-GM}{\le}\left(\frac{8x+y+8y+z+8z+x}{3}\right)^3=$$
$$=27(x+y+z)=27\big(using\ (1)\big)\ (3)$$

$$\prod\left(\frac{8}{r_a^2}+\frac{1}{r_ar_b}\right)=\prod\left(\frac{8x^2}{r^2}+\frac{xy}{r^2}\right)=\prod\frac{x}{r^2}(8x+y)=$$
$$=\frac{xyz}{r^6}(8x+y)(8y+z)(8z+x)\overset{(3)\&(2)}{\le}\frac{1}{r^6}\cdot\frac{1}{27}.27=\frac{1}{r^6}$$

$$Equality\ holds\ for\ an\ \ equilateral\ triangle.$$