ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\prod\left(\frac{8}{h_a^2}+\frac{1}{h_a h_b}\right)\le\frac{1}{r^6}$$

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$$Let\,\frac{r}{h_a}=x,\frac{r}{h_b}=y,\frac{r}{h_c}=z\ \ then\ x+y+z=r\sum\frac{1}{h_a}=1\ (1)$$

$$and\ xyz\overset{AM-GM}{\le}\left(\frac{x+y+z}{3}\right)^3=\frac{1}{27}\ \ (2)$$

$$(8x+y)(8y+z)(8z+x)\overset{AM-GM}{\le}\left(\frac{8x+y+8y+z+8z+x}{3}\right)^3=$$

$$=27(x+y+z)=27\big(using\ (1)\big)\ (3)$$

$$\prod\left(\frac{8}{h_a^2}+\frac{1}{h_a h_b}\right)=\prod\left(\frac{8x^2}{r^2}+\frac{xy}{r^2}\right)=\frac{xyz}{r^6}(8x+y)(8y+z)(8z+x)\overset{(2)\&(3)}{\le}$$

$$\le\frac{1}{r^6}\cdot\frac{27.1}{27}=\frac{1}{r^6}$$

Equality holds for $a=b=c$.