ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$6\le\sum\left(\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\right)\le\frac{3R}{r}$$

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Solution by Tapas Das-India

$$\sum\sqrt{\frac{h_a}{r_a}}\overset{CBS}{\le}\sqrt{\left(\sum h_a\right)\left(\sum\frac{1}{r_a}\right)}\overset{h_a\le m_a}{\le}\sqrt{\left(\sum m_a\right)\left(\sum\frac{1}{r_a}\right)}\overset{Leunberger}{\le}$$

$$\le\sqrt{(4R+r)\frac{1}{r}}\overset{Euler}{\le}\sqrt{\frac{9R}{2r}}=\sqrt{\frac{9R^2}{2Rr}}\overset{Euler}{\le}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

$$\sum\sqrt{\frac{r_a}{h_a}}\overset{CBS}{\le}\sqrt{\left(\sum r_a\right)\left(\sum\frac{1}{h_a}\right)}=\sqrt{(4R+r)\frac{1}{r}}\overset{Euler}{\le}\sqrt{\frac{9R}{2r}}=\sqrt{\frac{9R^2}{2Rr}}\overset{Euler}{\le}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

$$\sum\left(\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\right)=\sum\sqrt{\frac{h_a}{r_a}}+\sum\sqrt{\frac{r_a}{h_a}}\le\frac{3R}{2r}+\frac{3R}{2r}=\frac{3R}{r}\ and$$

$$\sum\left(\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\right)\overset{AM-GM}{\ge}2+2+2=6$$

Equality holds for $a=b=c$.