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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{4\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\le\frac{4R+r}{2s}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{s}{4r_a+r_b+r_c}=\sum_{cyc}\frac{s}{4R+r+3r_a}$$

$$=s*\frac{\sum_{cyc}(4R+r+3r_b)(4R+r+3r_c)}{(4R+r+3r_a)(4R+r+3r_b)(4R+r+3r_c)}$$

$$=s*\frac{3(4R+r)^2+6(4R+r)\left(\sum_{cyc}r_a\right)+9\sum_{cyc}r_br_c}{(4R+r)^3+9(4R+r)\left(\sum_{cyc}r_br_c\right)+3(4R+r)^2\left(\sum_{cyc}r_a\right)+27rs^2}$$

$$=s*\frac{3(4R+r)^2+6(4R+r)(4R+r)+9s^2}{(4R+r)^3+9(4R+r)s^2+3(4R+r)^2(4R+r)+27rs^2}\overset{?}{\le}\frac{4R+r}{2s}$$

$$\Leftrightarrow 2(4R+r)^4\underset{(*)}{\overset{?}{\ge}}(72R^2-18Rr-9r^2)s^2+9s^4$$

$$\text{Now, RHS of }(*)\overset{\text{Gerretsen}}{\le}(72R^2-18Rr-9r^2)s^2+9\left(4R^2+4Rr+3r^2\right)s^2$$

$$\overset{?}{\le}2(4R+r)^4\Leftrightarrow(4R+r)^4\underset{(**)}{\overset{?}{\ge}}(54R^2+9Rr+9r^2)s^2$$

$$\text{Again, RHS of }(**)\overset{\text{Gerretsen}}{\le}\left(54R^2+9Rr+9r^2\right)\left(4R^2+4Rr+3r^2\right)\overset{?}{\le}(4R+r)^4$$

$$\Leftrightarrow 40t^4+4t^3-138t^2-47t-26\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(40t^3+84t^2+30t+13\right)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\therefore(**)\Rightarrow(*)\text{ is true}\therefore$$

$$\sum_{cyc}\frac{1}{4\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\le\frac{4R+r}{2s}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := \frac{(4R+r)^2}{p^2} \overset{Doucet}{\geq} 3$. **Since we have** $\sum_{cyc} \tan\frac{A}{2} = \frac{4R+r}{p}$,

$$\sum_{cyc} \tan\frac{B}{2}\tan\frac{C}{2} = 1, \text{then}$$

$$\sum_{cyc} \frac{1}{4\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}} = \frac{1}{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}} \cdot \sum_{cyc}\left(1 - \frac{3\tan\frac{A}{2}}{4\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\right)$$

$$\overset{CBS}{\leq} \frac{p}{4R+r}\left(3 - \frac{3\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{\sum_{cyc}\tan\frac{A}{2}\left(4\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)}\right)$$

$$= \frac{p}{4R+r}\left(3 - \frac{3(4R+r)^2}{4(4R+r)^2-6p^2}\right)$$

$$= \frac{9p[(4R+r)^2-2p^2]}{2(4R+r)[2(4R+r)^2-3p^2]} \overset{?}{\leq} \frac{4R+r}{2p} \Leftrightarrow \frac{9(x-2)}{2x-3} \overset{?}{\leq} x \Leftrightarrow (x-3)^2 \geq 0,$$

which is true. Equality holds iff ΔABC **is equilateral.**