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In $\Delta ABC, O$ –circumcenter, r_1, r_2, r_3 –inradii of $\Delta OBC, \Delta OCA, \Delta OAB$.
Prove that:

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\ge\frac{6+4\sqrt{3}}{R}$$

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Known results:

$$\sum_{cyc}\frac{1}{\sin 2A}\ge\frac{9R^2}{2F}\ge\frac{s^2+(R+r)^2}{2F}\ge\frac{2(4R+r)}{s}\ge 2\sqrt{3}\qquad(1)$$

$$\sum_{cyc}\frac{1}{\cos A}=\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}\ge 6\qquad(2)$$

Back to the problem:

$$\sum_{cyc}\frac{1}{r_1}=\sum_{cyc}\frac{\frac{OB+OC+BC}{2}}{[OBC]}=\sum_{cyc}\frac{\frac{1}{2}(R+R+a)}{\frac{1}{2}\cdot R\cdot R\cdot\sin 2A}=\sum_{cyc}\frac{2R+a}{R^2\sin 2A}=$$

$$=\sum_{cyc}\frac{2R}{R^2\sin 2A}+\sum_{cyc}\frac{a}{R^2\sin 2A}=\frac{2}{R}\sum_{cyc}\frac{1}{\sin 2A}+\sum_{cyc}\frac{2R\sin A}{2R^2\sin A\cos A}\ge$$

$$\overset{(1)}{\ge}\frac{2}{R}\cdot 2\sqrt{3}+\frac{1}{R}\sum_{cyc}\frac{1}{\cos A}\overset{(2)}{\ge}\frac{4\sqrt{3}}{R}+\frac{1}{R}\cdot 6=\frac{6+4\sqrt{3}}{R}$$

Equality holds for $a=b=c$.