ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{xr_a + yr_b}{zr_c} + \frac{yr_a + zr_c}{xr_b} + \frac{xr_c + zr_b}{yr_a} \geq 12\sqrt[3]{\frac{r}{4R}}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$\frac{xr_a + yr_b}{zr_c} + \frac{yr_a + zr_c}{xr_b} + \frac{xr_c + zr_b}{yr_a} =$$

$$= \left(\frac{x}{z}\frac{r_a}{r_c} + \frac{y}{x}\frac{r_a}{r_b} + \frac{x}{y}\frac{r_c}{r_a}\right) + \left(\frac{y}{z}\frac{r_b}{r_c} + \frac{z}{x}\frac{r_c}{r_b} + \frac{z}{y}\frac{r_b}{r_a}\right) \overset{AM-GM}{\geq}$$

$$\geq 6\left(\left(\frac{x}{z}\frac{r_a}{r_c}\cdot\frac{y}{x}\frac{r_a}{r_b}\cdot\frac{x}{y}\frac{r_c}{r_a}\right)\cdot\left(\frac{y}{z}\frac{r_b}{r_c}\cdot\frac{z}{x}\frac{r_c}{r_b}\cdot\frac{z}{y}\frac{r_b}{r_a}\right)\right)^{\frac{1}{3}} =$$

$$= 6(1)^{\frac{1}{3}} = 6 = 12\cdot\frac{1}{2} = 12\left(\frac{1}{8}\right)^{\frac{1}{3}} = 12\left(\frac{r}{8r}\right)^{\frac{1}{3}} \overset{Euler}{\geq} 12\left(\frac{r}{\frac{8R}{2}}\right)^{\frac{1}{3}} = 12\sqrt[3]{\frac{r}{4R}}$$