ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$3\sqrt{3}r \le \frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b} \le \left(\frac{R}{r}-1\right)s$$

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$$WLOG\ a \ge b \ge c\ then\ r_a \ge r_b \ge r_c\ and\ (r_a+r_b) \ge (r_a+r_c) \ge (r_b+r_c)$$
$$and\ \frac{r_a}{(r_b+r_c)} \ge \frac{r_b}{(r_a+r_c)} \ge \frac{r_c}{(r_a+r_b)}$$

$$\sum ar_a = F\sum \frac{a}{s-a} = \frac{F}{(s-a)(s-b)(s-c)}\left(\sum a(s-b)(s-c)\right) =$$

$$= \frac{F}{sr^2}(s^2(a+b+c)-2s(ab+bc+ca)+3abc) =$$

$$= \frac{F}{sr^2}(2s^3-2s(s^2+r^2+4Rr)+12Rrs) = \frac{F}{sr^2}(2Rr-r^2)2s = \frac{2s(2Rr-r^2)}{r} \quad (1)$$

$$\frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b} = \sum \frac{ar_a}{r_b+r_c} \overset{AM-HM}{\le} \sum \frac{1}{4}a\left(\frac{r_a}{r_b}+\frac{r_a}{r_c}\right) =$$

$$\sum \frac{1}{4}a\left(\frac{r_a}{r_b}+\frac{r_a}{r_c}+\frac{r_a}{r_a}-1\right) = \sum ar_a\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\frac{1}{4} - \sum \frac{a}{4} =$$

$$= \frac{1}{4r}\sum ar_a - \frac{s}{2} \overset{(1)}{=} \frac{2s(2Rr-r^2)}{r(4r)} - \frac{s}{2} = \frac{s}{2}\left(\frac{2R}{r}-2\right) = \left(\frac{R}{r}-1\right)s$$

$$\frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b} \overset{Chebyshev}{\ge}$$

$$\ge \frac{1}{3}\left(\sum a\right)\left(\sum \frac{r_a}{r_b+r_c}\right) \overset{Nesbitt}{\ge} \frac{1}{3}2s \cdot \frac{3}{2} \overset{Mitrinovic}{\ge} 3\sqrt{3}r$$

Equality holds for $a=b=c$.