ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{9}{4s^2} \le \frac{1}{(r_a+h_a)^2} + \frac{1}{(r_b+h_b)^2} + \frac{1}{(r_c+h_c)^2} \le \frac{1}{12r^2}$$

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$$\sum_{cyc}\frac{a(s-a)}{b+c} = \sum_{cyc}\frac{(a-2s+2s)(s-a)}{2s-a} =$$

$$= -\sum_{cyc}(s-a) + 2s^2\sum_{cyc}\frac{1}{b+c} + 2s\sum_{cyc}\frac{-a+2s-2s}{2s-a}$$

$$= -s + 2s^2\sum_{cyc}\frac{1}{b+c} + 6s - 4s^2\sum_{cyc}\frac{1}{b+c} = 5s - 2s^2.\sum_{cyc}\frac{(c+a)(a+b)}{\prod_{cyc}(b+c)}$$

$$= 5s - 2s^2.\frac{\left(\sum_{cyc}a^2 + 2\sum_{cyc}ab\right) + \sum_{cyc}ab}{2s(s^2+2Rr+r^2)} = 5s - 2s^2.\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}$$

$$= 5s - s.\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2} \Rightarrow \sum_{cyc}\frac{a(s-a)}{b+c} = \frac{s(6Rr+4r^2)}{s^2+2Rr+r^2} \rightarrow (1)$$

$$\text{Now, } \sum_{cyc}\frac{1}{(r_a+h_a)^2} = \sum_{cyc}\frac{1}{\left(\frac{rs}{s-a}+\frac{2rs}{a}\right)^2} = \frac{1}{r^2s^2}.\sum_{cyc}\frac{a^2(s-a)^2}{(b+c)^2}$$

$$\ge \frac{1}{3r^2s^2}.\left(\sum_{cyc}\frac{a(s-a)}{b+c}\right)^2 \overset{\text{via }(1)}{\ge} \frac{s^2r^2(6R+4r)^2}{3r^2s^2(s^2+2Rr+r^2)^2} \overset{?}{\ge} \frac{9}{4s^2}$$

$$\Leftrightarrow 4s^2(6R+4r)^2 \overset{?}{\ge} 27(s^2+2Rr+r^2)^2$$

$$\Leftrightarrow (144R^2+84Rr+10r^2)s^2 \underbrace{\overset{?}{\ge}}_{(*)} 27s^4 + 27r^2(2R+r)^2$$

$$\text{Now, RHS of }(*) \overset{\text{Gerretsen}}{\le} (108R^2+108Rr+81r^2)s^2 + 27r^2(2R+r)^2$$

$$\overset{?}{\le} (144R^2+84Rr+10r^2)s^2 \Leftrightarrow (36R^2-24Rr-71r^2)s^2 \underbrace{\overset{?}{\ge}}_{(**)} 27r^2(2R+r)^2$$

$$\text{Again, } \because 36R^2-24Rr-71r^2 = (R-2r)(36R+48r) + 25r^2 \overset{\text{Euler}}{\ge} 25r^2 > 0$$

$$\therefore \text{LHS of }(**) \overset{\text{Gerretsen}}{\ge} (36R^2-24Rr-71r^2)(16Rr-5r^2) \overset{?}{\ge} 27r^2(2R+r)^2$$

$$\Leftrightarrow 144t^3 - 168t^2 - 281t + 82 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(144t^2+99t+21(t-2)+1) \overset{?}{\ge} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{is true} \therefore \sum_{cyc} \frac{1}{(r_a + h_a)^2} \geq \frac{9}{4s^2}$$

$$Also, \sum_{cyc} \frac{1}{(r_a + h_a)^2} \overset{A-G}{\leq} \sum_{cyc} \frac{1}{4r_a h_a} = \sum_{cyc} \frac{a(s-a)}{8r^2 s^2}$$

$$= \frac{1}{8r^2 s^2} \cdot \left(s(2s) - 2(s^2 - 4Rr - r^2)\right) = \frac{4R + r}{4rs^2} \overset{\text{Gerretsen + Euler}}{\leq} \frac{4R + r}{4r.3r(4R + r)}$$

$$\therefore \sum_{cyc} \frac{1}{(r_a + h_a)^2} \leq \frac{1}{12r^2} \text{ and so,}$$

$$\frac{9}{4s^2} \leq \frac{1}{(r_a + h_a)^2} + \frac{1}{(r_b + h_b)^2} + \frac{1}{(r_c + h_c)^2} \leq \frac{1}{12r^2} \ \forall \Delta ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)