ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\cdot\frac{a}{b+c}+\frac{y+z}{x}\cdot\frac{b}{c+a}+\frac{z+x}{y}\cdot\frac{c}{a+b}\geq\frac{6}{\sqrt[3]{4+\frac{2R}{r}}}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$\frac{abc}{(a+b)(b+c)(c+a)}=\frac{4Rrs}{(a+b+c)(ab+bc+ca)-abc}=$$

$$=\frac{4Rrs}{2s(s^2+r^2+4Rr)-4Rrs}=\frac{2Rr}{s^2+r^2+4Rr-2Rr}=$$

$$=\frac{2Rr}{s^2+r^2+2Rr}\overset{Gerretsen}{\geq}\frac{2Rr}{4R^2+4Rr+3r^2+r^2+2Rr}=$$

$$=\frac{2Rr}{4R^2+6Rr+4r^2}=\frac{1}{\frac{2R}{r}+3+\frac{2r}{R}}\overset{Euler}{\geq}\frac{1}{\frac{2R}{r}+3+1}=\frac{1}{4+\frac{2R}{r}}\quad(1)$$

$$\frac{x+y}{z}\frac{a}{b+c}+\frac{y+z}{x}\frac{b}{c+a}+\frac{z+x}{y}\frac{c}{a+b}=$$

$$=\sum\frac{x+y}{z}\frac{a}{b+c}\overset{AM-GM}{\geq}2\sum\frac{\sqrt{xy}}{z}\frac{a}{b+c}\overset{AM-GM}{\geq}$$

$$\geq 6\left(\frac{abc}{(a+b)(b+c)(c+a)}\right)^{\frac{1}{3}}\overset{(1)}{\geq}\frac{6}{\sqrt[3]{4+\frac{2R}{r}}}$$

Equality holds for $x=y=z, a=b=c$.