ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda, \mu > 0$ then in ΔABC the following relationship holds:

$$4\sqrt{3}(\lambda+\mu)\cdot\frac{r}{R} \le \sum_{cyc}\frac{\lambda a+\mu b}{r_c} \le \frac{3R(\lambda+\mu)}{F}\cdot\sqrt{9R^2-s^2}$$

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$$\sum_{cyc}\frac{\lambda a+\mu b}{r_c} = \lambda\sum\frac{a}{r_c}+\mu\sum\frac{b}{r_c} = \frac{\lambda}{F}\sum a(s-c)+\frac{\mu}{F}\sum b(s-c) =$$

$$= \frac{\lambda}{F}\left(2s^2-\sum ac\right)+\frac{\mu}{F}\left(2s^2-\sum bc\right) =$$

$$= \frac{\lambda+\mu}{F}\left(2s^2-\sum bc\right) = \frac{\lambda+\mu}{F}(2s^2-s^2-r^2-4Rr) =$$

$$= \frac{\lambda+\mu}{F}(s^2-r^2-4Rr) = \frac{\lambda+\mu}{2F}\,2(s^2-r^2-4Rr) =$$

$$= \frac{\lambda+\mu}{2F}\left(\sum a^2\right) \overset{Leibniz}{\le} \frac{\lambda+\mu}{2F}(9R^2)$$

$$We\ need\ to\ show\ \frac{\lambda+\mu}{2F}(9R^2) \le \frac{3R(\lambda+\mu)}{F}.\sqrt{9R^2-s^2}\ or$$

$$\frac{3R}{2} \le \sqrt{9R^2-s^2}\ or, \frac{9R^2}{4} \le 9R^2-s^2$$

$$or\ s^2 \le \frac{27R^2}{4}\ true\ (Mitrinovic)$$

$$Again\ from\ the\ previous\ result:$$

$$\sum_{cyc}\frac{\lambda a+\mu b}{r_c} = \frac{\lambda+\mu}{2F}\left(\sum a^2\right) \overset{Ionescu-Weitzenbock}{\ge}$$

$$\ge \frac{\lambda+\mu}{2F}4\sqrt{3}\,F = 4\sqrt{3}(\lambda+\mu).\frac{1}{2} \overset{Euler}{\ge} 4\sqrt{3}(\lambda+\mu)\frac{r}{R}$$