ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC, I$ −incenter, IM, IN, IP −medians in $\Delta IBC, \Delta ICA, \Delta IAB$

$M \in (BC), N \in (CA), P \in (AB), IM = v_a, IN = v_b, IP = v_c$. Prove that:

$$2Rr - r^2 \le v_a^2 + v_b^2 + v_c^2 \le 2R^2 - 4Rr + 3r^2$$

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Solution by Daniel Sitaru-Romania

$$\sum_{cyc} v_a^2 = \sum_{cyc} \frac{1}{2}(IB^2 + IC^2) - \frac{1}{4}\sum_{cyc} a^2 =$$

$$= \sum_{cyc} IA^2 - \frac{1}{4}\cdot 2(s^2 - r^2 - 4Rr) = \sum_{cyc} \frac{r^2}{\sin^2\frac{A}{2}} - \frac{1}{2}(s^2 - r^2 - 4Rr) =$$

$$= r^2 \sum_{cyc} \frac{1}{\sin^2\frac{A}{2}} - \frac{1}{2}(s^2 - r^2 - 4Rr) = r^2 \cdot \frac{s^2 + r^2 - 8Rr}{r^2} - \frac{s^2}{2} + \frac{r^2}{2} + 2Rr$$

$$= s^2 + r^2 - 8Rr - \frac{s^2}{2} + \frac{r^2}{2} + 2Rr = \frac{s^2}{2} + \frac{3r^2}{2} - 6Rr$$

$$\sum_{cyc} v_a^2 = \frac{s^2}{2} + \frac{3r^2}{2} - 6Rr \overset{GERRETSEN}{\le} \frac{1}{2}(4R^2 + 4Rr + 3r^2) + \frac{3r^2}{2} - 6Rr =$$

$$= 2R^2 - 4Rr + 3r^2$$

$$\sum_{cyc} v_a^2 = \frac{s^2}{2} + \frac{3r^2}{2} - 6Rr \overset{GERRETSEN}{\ge} \frac{1}{2}(16Rr - 5r^2) + \frac{3r^2}{2} - 6Rr =$$

$$= 2Rr - r^2$$

Equality holds for $a = b = c$.