ROMANIAN MATHEMATICAL MAGAZINE

G –centroid of ΔABC, $A', B', C' \in Ext(\Delta ABC)$, (G, A, A'), (G, B, B'), (G, C, C') –collinears, $AA' = BC, BB' = CA, CC' = AB$. Prove that:

$$\frac{[A'B'C']}{[ABC]} \geq \left(1 + \frac{2r}{R}\right)^2$$

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Solution by Daniel Sitaru-Romania

$$GA = \frac{2}{3}m_a, GB = \frac{2}{3}m_b, GC = \frac{2}{3}m_c$$

$$cos(\measuredangle AGB) = \frac{\left(\frac{2}{3}m_a\right)^2 + \left(\frac{2}{3}m_b\right)^2 - c^2}{2 \cdot \frac{2}{3}m_a \cdot \frac{2}{3}m_b}$$

$$GA' = \frac{4}{3}m_a, GB' = \frac{4}{3}m_b, GC' = \frac{4}{3}m_c$$

$$A'B'^2 = \left(\frac{4}{3}m_a\right)^2 + \left(\frac{4}{3}m_b\right)^2 - 2 \cdot \frac{4}{3}m_a \cdot \frac{4}{3}m_b \cdot cos(\measuredangle AGB)$$

$$A'B'^2 = \left(\frac{4}{3}m_a\right)^2 + \left(\frac{4}{3}m_b\right)^2 - 2 \cdot \frac{4}{3}m_a \cdot \frac{4}{3}m_b \cdot \frac{\left(\frac{2}{3}m_a\right)^2 + \left(\frac{2}{3}m_b\right)^2 - c^2}{2 \cdot \frac{2}{3}m_a \cdot \frac{2}{3}m_b}$$

$$A'B'^2 = \left(\frac{4}{3}m_a\right)^2 + \left(\frac{4}{3}m_b\right)^2 - 4\left(\left(\frac{2}{3}m_a\right)^2 + \left(\frac{2}{3}m_b\right)^2\right) + 4c^2$$

$$A'B'^2 = 4c^2 \Rightarrow A'B' = 2c, B'C' = 2a, C'A' = 2b$$

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = 2 \Rightarrow \frac{[A'B'C']}{[ABC]} = 2^2 = 4$$

$$4 \geq \left(1 + \frac{2r}{R}\right)^2 \Leftrightarrow 2 \geq 1 + \frac{2r}{R} \Leftrightarrow R \geq 2r \text{ (Euler)}$$

Equality holds for $a = b = c$.