ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a}{h_a - r} + \frac{r_b}{h_b - r} + \frac{r_c}{h_c - r} \ge \frac{9}{2}$$

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$$\frac{r_a}{h_a - r} + \frac{r_b}{h_b - r} + \frac{r_c}{h_c - r} = \sum_{cyc} \frac{r_a}{h_a - r} =$$

$$= \sum_{cyc} \frac{\frac{F}{s-a}}{\frac{2F}{a} - \frac{F}{s}} = \sum_{cyc} \frac{\frac{1}{s-a}}{\frac{2}{a} - \frac{1}{s}} = \sum_{cyc} \frac{sa}{(2s-a)(s-a)} = s\sum_{cyc} \frac{a}{(b+c)(s-a)} \ge$$

$$\overset{AM-GM}{\ge} 3s \cdot \sqrt[3]{\frac{abc}{(s-a)(s-b)(s-c)(a+b)(b+c)(c+a)}} =$$

$$= 3s \cdot \sqrt[3]{\frac{4RFs}{s(s-a)(s-b)(s-c)}} \cdot \frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}} \overset{AM-GM}{\ge}$$

$$= 3s \cdot \sqrt[3]{\frac{4RFs}{F^2}} \cdot \frac{1}{\frac{a+b+b+c+c+a}{3}} = 3s \cdot \sqrt[3]{\frac{4Rs}{F}} \cdot \frac{3}{4s} =$$

$$= 9 \cdot \sqrt[3]{\frac{4Rs}{rs}} \cdot \frac{1}{4} = \frac{9}{4} \cdot \sqrt[3]{\frac{4R}{r}} \overset{EULER}{\ge} \frac{9}{4} \cdot \sqrt[3]{\frac{8r}{r}} = \frac{9}{2}$$

Equality holds for $a = b = c$.