ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a-r}{h_a-r}+\frac{r_b-r}{h_b-r}+\frac{r_c-r}{h_c-r}\ge 3$$

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Solution by Daniel Sitaru-Romania

$$\frac{r_a-r}{h_a-r}+\frac{r_b-r}{h_b-r}+\frac{r_c-r}{h_c-r}=\sum_{cyc}\frac{r_a-r}{h_a-r}=$$

$$=\sum_{cyc}\frac{\frac{F}{s-a}-\frac{F}{s}}{\frac{2F}{a}-\frac{F}{s}}=\sum_{cyc}\frac{\frac{1}{s-a}-\frac{1}{s}}{\frac{2}{a}-\frac{1}{s}}=\sum_{cyc}\frac{as(s-s+a)}{s(2s-a)(s-a)}=$$

$$=\sum_{cyc}\frac{a^2}{(s-a)(b+c)}\overset{AM-GM}{\ge}3\cdot\sqrt[3]{\frac{(abc)^2}{(s-a)(s-b)(s-c)(a+b)(b+c)(c+a)}}=$$

$$=3\cdot\sqrt[3]{\frac{(abc)^2\cdot s}{s(s-a)(s-b)(s-c)}}\cdot\frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}}\overset{AM-GM}{\ge}$$

$$\ge 3\cdot\sqrt[3]{\frac{(4RF)^2\cdot s}{F^2}}\cdot\frac{3}{a+b+b+c+c+a}=\frac{9}{4s}\cdot\sqrt[3]{16R^2s}\ge$$

$$\overset{MITRINOVIC}{\ge}\frac{9}{4s}\cdot\sqrt[3]{16s\cdot\frac{4}{27}s^2}=\frac{9}{4s}\cdot\frac{4s}{3}=3$$

Equality holds for $a=b=c$.