ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sqrt[3]{m_a m_b m_c} + \frac{|m_a - m_b| + |m_b - m_c| + |m_c - m_a|}{2} \geq \frac{m_a + m_b + m_c}{3}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, we shall prove : } \frac{|a-b|+|b-c|+|c-a|}{2} \overset{(1)}{\geq} \frac{a+b+c}{3} - \sqrt[3]{abc}$$

$$\text{Now, } \frac{a+b+c}{3} - \sqrt[3]{abc} \overset{G-H}{\leq} \frac{\sum_{cyc} a}{3} - \frac{3abc}{\sum_{cyc} ab}$$

$$= \frac{(\sum_{cyc} a)(\sum_{cyc} ab) - 9abc}{3\sum_{cyc} ab} \overset{?}{\leq} \frac{|a-b|+|b-c|+|c-a|}{2}$$

$$\Leftrightarrow \sum_{cyc} (b-c)^2 + 2\sum_{cyc} (|a-b||b-c|) \underset{(*)}{\overset{?}{\geq}} \frac{4}{9}\left(\frac{(\sum_{cyc} a)(\sum_{cyc} ab) - 9abc}{\sum_{cyc} ab}\right)^2$$

$$\text{Now, LHS of } (*) \geq 2\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) \overset{?}{\geq} \frac{4}{9}\left(\frac{(\sum_{cyc} a)(\sum_{cyc} ab) - 9abc}{\sum_{cyc} ab}\right)^2$$

$$\Leftrightarrow 9(s^2 - 12Rr - 3r^2)(s^2 + 4Rr + r^2)^2 \overset{?}{\geq} 2(2s(s^2 + 4Rr + r^2) - 36Rrs)^2$$

$$\Leftrightarrow s^6 + (188Rr - 25r^2)s^4 - r^2s^2(2288R^2 + 136Rr + 53r^2) - 27r^3(4R + r)^3 \underset{(**)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (*) \overset{Gerretsen}{\geq} (204Rr - 30r^2)s^4 - r^2s^2(2288R^2 + 136Rr + 53r^2)$$

$$-27r^3(4R + r)^3 \overset{Gerretsen}{\geq} \begin{pmatrix} (204Rr - 30r^2)(16Rr - 5r^2) \\ -r^2(2288R^2 + 136Rr + 53r^2) \end{pmatrix} s^2 - 27r^3(4R + r)^3$$

$$= r^2(976R^2 - 1636Rr + 97r^2)s^2 - 27r^3(4R + r)^3 \overset{Gerretsen}{\geq}$$

$$r^3(976R^2 - 1636Rr + 97r^2)(16R - 5r) - 27r^3(4R + r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow 434t^3 - 1011t^2 + 294t - 16 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)(362t^2 + 72t(t - 2) + t + 8) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a+b+c}{3} - \sqrt[3]{abc} \leq \frac{|a-b|+|b-c|+|c-a|}{2} \Rightarrow (1) \text{ is true and invoking } (1)$$

on a triangle with sides m_a, m_b, m_c, we arrive at :

$$\frac{|m_a - m_b| + |m_b - m_c| + |m_c - m_a|}{2} \geq \frac{m_a + m_b + m_c}{3} - \sqrt[3]{m_a m_b m_c}$$

$$\Rightarrow \sqrt[3]{m_a m_b m_c} + \frac{|m_a - m_b| + |m_b - m_c| + |m_c - m_a|}{2} \geq \frac{m_a + m_b + m_c}{3}$$

$\forall \Delta$ ABC, "=" iff Δ ABC is equilateral (QED)