ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\left(\sum_{cyc}\frac{a}{a+b}\right)\left(\sum_{cyc}\frac{a}{b+c}\right)+\left(\frac{R}{2r}\right)^3\geq 1+\left(\sum_{cyc}\frac{m_a}{m_a+m_b}\right)\left(\sum_{cyc}\frac{m_a}{m_b+m_c}\right)$$

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Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

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$$\left(\sum_{cyc}\frac{a}{a+b}\right)\left(\sum_{cyc}\frac{a}{b+c}\right)+\left(\frac{R}{2r}\right)^3-1\overset{\text{Nesbitt}}{\geq}\left(\sum_{cyc}\frac{a^2}{a^2+ab}\right)\left(\frac{3}{2}\right)+\frac{R^3-8r^3}{8r^3}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{4s^2}{2(s^2-4Rr-r^2)+s^2+4Rr+r^2}\cdot\frac{3}{2}+\frac{R^3-8r^3}{8r^3}$$

$$\Rightarrow\left(\sum_{cyc}\frac{a}{a+b}\right)\left(\sum_{cyc}\frac{a}{b+c}\right)+\left(\frac{R}{2r}\right)^3-1\overset{(*)}{\geq}\frac{(R^3-8r^3)(3s^2-4Rr-r^2)+48r^3s^2}{8r^3(3s^2-4Rr-r^2)}$$

$$\text{Again, }\left(\sum_{cyc}\frac{m_a}{m_a+m_b}\right)\left(\sum_{cyc}\frac{m_a}{m_b+m_c}\right)\overset{\text{CBS}}{\leq}$$

$$\sqrt{\sum_{cyc}m_a^2}\cdot\sqrt{\sum_{cyc}\frac{1}{(m_a+m_b)^2}}\cdot\sqrt{\sum_{cyc}m_a^2}\cdot\sqrt{\sum_{cyc}\frac{1}{(m_b+m_c)^2}}$$

$$=\frac{3}{4}\cdot 2(s^2-4Rr-r^2)\cdot\sum_{cyc}\frac{1}{(m_b+m_c)^2}\overset{A-G}{\leq}\frac{3}{2}\cdot(s^2-4Rr-r^2)\sum_{cyc}\frac{m_a}{4m_am_bm_c}$$

$$\overset{\substack{\text{Leuenberger}\\ \text{and}\\ \text{Lascu}}}{\leq}\frac{3}{8}\cdot(s^2-4Rr-r^2)\cdot\frac{(4R+r)}{\prod_{cyc}\left(\frac{b+c}{2}\cos\frac{A}{2}\right)}$$

$$=\frac{3}{8}\cdot(s^2-4Rr-r^2)\cdot\frac{8(4R+r).4R}{2s^2(s^2+2Rr+r^2)}$$

$$\Rightarrow\left(\sum_{cyc}\frac{m_a}{m_a+m_b}\right)\left(\sum_{cyc}\frac{m_a}{m_b+m_c}\right)\overset{(**)}{\leq}\frac{6R(4R+r)(s^2-4Rr-r^2)}{s^2(s^2+2Rr+r^2)}\;\therefore\;(*),(**)\Rightarrow$$

$$\text{it suffices to prove : }\frac{(R^3-8r^3)(3s^2-4Rr-r^2)+48r^3s^2}{8r^3(3s^2-4Rr-r^2)}$$

$$\geq\frac{6R(4R+r)(s^2-4Rr-r^2)}{s^2(s^2+2Rr+r^2)}$$

$$\Leftrightarrow \boxed{\begin{gathered}(3R^3+24r^3)s^6+r(2R^4+2R^3r-576R^2r^2-64Rr^3+32r^4)s^4-\\ r^2(8R^5+6R^4r-3071R^3r^2-1600R^2r^3-240Rr^4-8r^5)s^2-48Rr^5(4R+r)^3\overset{(\bullet)}{\ge}0\end{gathered}}$$

$$\because (3R^3+24r^3)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (\bullet),$$

$$\text{it suffices to prove : LHS of } (\bullet) \ge (3R^3+24r^3)(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (146R^4-43R^3r-576R^2r^2+1088Rr^3-328r^4)s^4$$

$$-r(2312R^5-1434R^4r-2846R^3r^2+16832R^2r^3-11760Rr^4+1792r^5)s^2$$

$$+r^2\begin{pmatrix}12288R^6-11520R^5r+528R^4r^2+95625R^3r^3\\ -92736R^2r^4+28752Rr^5-3000r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet)}{\ge}0}$$

$$\text{Now, } 146R^4-43R^3r-576R^2r^2+1088Rr^3-328r^4$$

$$=(R-2r)(146R^3+249R^2r-78Rr^2+932r^3)+1536r^4 \overset{\text{Euler}}{\ge} 1536r^4>0 \therefore \text{LHS}$$

$$\text{of } (\bullet\bullet) \overset{\text{Gerretsen}}{\ge} (146R^4-43R^3r-576R^2r^2+1088Rr^3-328r^4)(16Rr-5r^2)s^2$$

$$-r(2312R^5-1434R^4r-2846R^3r^2+16832R^2r^3-11760Rr^4+1792r^5)s^2$$

$$+r^2\begin{pmatrix}12288R^6-11520R^5r+528R^4r^2+95625R^3r^3\\ -92736R^2r^4+28752Rr^5-3000r^6\end{pmatrix}\overset{?}{\ge}0$$

$$\Leftrightarrow \boxed{\begin{gathered}(24R^5+16R^4r-6155R^3r^2+3456R^2r^3+1072Rr^4-152r^5)s^2+\\ r(12288R^6-11520R^5r+528R^4r^2+95625R^3r^3-92736R^2r^4+28752Rr^5-3000r^6)\\ \overset{(\bullet\bullet\bullet)}{\ge}0\end{gathered}}$$

$\boxed{\textit{Case 1}}$ $24R^5+16R^4r-6155R^3r^2+3456R^2r^3+1072Rr^4-152r^5 \ge 0$ *and*

$$\text{then : LHS of } (\bullet\bullet\bullet) \overset{\text{Euler}}{\ge} r\begin{pmatrix}12288R^6-11520R^5r+528R^4r^2+\\ 95625R^3r^3-92736R^2r^4+28752Rr^5-3000r^6\end{pmatrix}$$

$$=(R-2r)\begin{pmatrix}12288R^5+13056R^4r+26640R^3r^2+148905r^3\\ +205074Rr^4+438900r^5\end{pmatrix}+874800r^6$$

$$\overset{\text{Euler}}{\ge} 874800r^6>0 \therefore (\bullet\bullet\bullet) \text{ is true}$$

$\boxed{\textit{Case 2}}$ $24R^5+16R^4r-6155R^3r^2+3456R^2r^3+1072Rr^4-152r^5<0$ *and*

$$\text{then : LHS of } (\bullet\bullet\bullet)$$

$$=-\left(-(24R^5+16R^4r-6155R^3r^2+3456R^2r^3+1072Rr^4-152r^5)\right)s^2+$$

$$r\begin{pmatrix}12288R^6-11520R^5r+528R^4r^2+95625R^3r^3\\ -92736R^2r^4+28752Rr^5-3000r^6\end{pmatrix}\overset{\text{Gerretsen}}{\ge}$$

$$-\left(-(24R^5+16R^4r-6155R^3r^2+3456R^2r^3+1072Rr^4-152r^5)\right)\begin{pmatrix}4R^2+4Rr\\ +3r^2\end{pmatrix}$$

$$+r\begin{pmatrix}12288R^6-11520R^5r+528R^4r^2+95625R^3r^3\\ -92736R^2r^4+28752Rr^5-3000r^6\end{pmatrix}\overset{?}{\ge}0 \Leftrightarrow$$

$$24t^7+3112t^6-9001t^5-2555t^4+23818t^3-19672t^2+7840t-864\overset{?}{\ge}0$$

$$\left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}24t^5+3208t^4+3535t^3\\ -447t^2+7090t+10476\end{pmatrix}+21384\right)\overset{?}{\ge}0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\bullet\bullet\bullet) \text{ is true } \textit{and} \text{ combining both } \textit{cases}, (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$

$$\Rightarrow (\bullet) \text{ is true } \forall \Delta ABC \therefore \left(\sum_{cyc}\frac{a}{a+b}\right)\left(\sum_{cyc}\frac{a}{b+c}\right)+\left(\frac{R}{2r}\right)^3 \ge$$

$$1+\left(\sum_{cyc}\frac{m_a}{m_a+m_b}\right)\left(\sum_{cyc}\frac{m_a}{m_b+m_c}\right) \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equi}\textit{lateral} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc}\frac{a}{a+b}\right)\left(\sum_{cyc}\frac{a}{b+c}\right)+\left(\frac{R}{2r}\right)^3\geq 1+\left(\sum_{cyc}\frac{m_a}{m_a+m_b}\right)\left(\sum_{cyc}\frac{m_a}{m_b+m_c}\right)\qquad (*)$$

Let $p:=a+b+c$ and $q:=ab+bc+ca$. By CBS inequality, we have

$$\left(\sum_{cyc}\frac{a}{a+b}\right)\left(\sum_{cyc}\frac{a}{b+c}\right)\geq\frac{(a+b+c)^2}{\sum_{cyc}a(a+b)}\cdot\frac{(a+b+c)^2}{\sum_{cyc}a(b+c)}=\frac{p^4}{(p^2-q).2q}$$

$$=\frac{9}{4}+\frac{(p^2-3q)(2p^2-3q)}{2q(p^2-q)}\overset{p^2\geq 3q}{\geq}\frac{9}{4}.$$

Now, we have

$$\bullet\sum_{cyc}\frac{m_a}{m_a+m_b}\overset{AM-GM}{\leq}\sum_{cyc}\frac{1}{2}\sqrt{\frac{m_a}{m_b}}\overset{CBS}{\leq}\frac{1}{2}\sqrt{\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_b}\right)}\overset{\substack{Leuenberger\\ m_a\geq h_a}}{\leq}\frac{1}{2}\sqrt{(4R+r)\left(\sum_{cyc}\frac{1}{h_a}\right)}=\sqrt{\frac{4R+r}{4r}}.$$

$$\bullet\sum_{cyc}\frac{m_a}{m_b+m_c}\overset{CBS}{\leq}\frac{1}{4}\sum_{cyc}\left(\frac{m_a}{m_b}+\frac{m_a}{m_c}\right)\overset{CBS}{\leq}\frac{1}{4}.2\sqrt{\left(\sum_{cyc}{m_a}^2\right)\left(\sum_{cyc}\frac{1}{{m_b}^2}\right)}\overset{m_a\geq\sqrt{s(s-a)}}{\leq}\frac{1}{2}\sqrt{\left(\sum_{cyc}{m_a}^2\right)\left(\sum_{cyc}\frac{1}{s(s-a)}\right)}$$

$$=\frac{1}{2}\sqrt{\frac{3(s^2-r^2-4Rr)(4R+r)}{2s^2r}}\overset{Gerretsen}{\leq}\frac{1}{2}\sqrt{\frac{3(4R^2+2r^2)(4R+r)}{2(16Rr-5r^2)r}}.$$

Using these results, we have

$$RHS_{(*)}\leq\frac{4R+r}{4r}\sqrt{\frac{3(2R^2+r^2)}{16Rr-5r^2}}+1\overset{AM-GM}{\leq}\frac{1}{8}\left(\frac{(4R+r)^2}{9r^2}+\frac{27(2R^2+r^2)}{16Rr-5r^2}\right)+1$$

$$=\left(\left(\frac{R}{2r}\right)^3+\frac{9}{4}\right)-\frac{(R-2r)[(R-2r)(144R^2+275Rr-10r^2)+324r^3]}{72(16Rr-5r^2)r^2}\overset{Euler}{\leq}$$

$$\leq\left(\frac{R}{2r}\right)^3+\frac{9}{4}\leq LHS_{(*)},$$

which completes the proof. Equality holds iff ΔABC is equilateral.